Section 4.7

Variation of Parameters

METHOD OF VARIATION OF PARAMETERS

For a second-order linear equation in standard form $y'' + Py' + Qy = f(x)$.

1. Find the complementary function $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$.
2. Replace the constants by the functions u_1 and u_2 to form the particular solution $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.
3. Solve the following system of equations for u_1' and u_2'

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(x) \end{aligned}$$

4. Integrate to find $u_1(x)$ and $u_2(x)$.

USING CRAMER'S RULE TO SOLVE THE SYSTEM

The system of equations

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_2' + u_2'y_2' &= f(x) \end{aligned}$$

can be solved using Cramer's Rule (determinants).

$$u_1' = \frac{W_1}{W} \text{ and } u_2' = \frac{W_2}{W} \text{ where}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \text{ and } W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

COMMENTS ON VARIATION OF PARAMETERS

1. The system of equations can be solved by other methods as well. That is, by substitution or elimination.
2. The method of Variation of Parameters is not limited to $f(x)$ being either a polynomial, exponential, sine, cosine, or finite sums and products of these functions.

HIGHER-ORDER EQUATIONS

Variation of Parameters can be used to solve higher-order equations. Let

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x)$$

Solve the following system of equations.

$$\begin{aligned} y_1u_1' + \quad y_2u_2' + \cdots + \quad y_nu_n' &= 0 \\ y_2'u_1' + \quad y_2'u_2' + \cdots + \quad y_n'u_n' &= 0 \\ &\vdots \\ y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' &= f(x) \end{aligned}$$