Section 4.5

Differential Operators

DIFFERENTIAL OPERATOR

In calculus, differentiation is often denoted by the capital letter *D*; that is,

$$\frac{dy}{dx} = Dy$$

The symbol *D* is called a **differential operator**, it transforms a differentiable function into another function.

The differential operator is a **linear** operator.

HIGHER-ORDER DERIVATIVES

Higher order derivatives can be expressed in terms of the differential operator.

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(Dy) = D^2y$$

In general,

$$y^{(n)} = \frac{d^ny}{dx^n} = D^ny$$

POLYNOMIAL EXPRESSIONS AND DIFFERENTIAL OPERATORS

Polynomial expressions involving *D* are also linear differential operators.

EXAMPLES:

$D + 3$
$D^2 + 3D - 4$
$5D^3 - 6D^2 + 4D + 9$

WRITING A DIFFERENTIAL EQUATION IN OPERATOR NOTATION

Differential equations can be written in operator notation.

ANNIHILATOR OPERATOR

If L is a linear differential operator with constant coefficients and $y = f(x)$ is a sufficiently differentiable function such that

$$L(y) = 0,$$

then *L* is said to be an **annihilator** of the function.

ANNIHILATOR FOR POLYNOMIALS

If

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

Then $D^{n+1}(y) = 0$ and, consequently, D^{n+1} annihilates y.

ANNIHILATORS FOR $e^{\alpha x}$

If

$$y = e^{\alpha x}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
$$= a_n e^{\alpha x} x^n + a_{n-1} e^{\alpha x} x^{n-1} + \cdots + a_1 e^{\alpha x} x + a_0 e^{\alpha x}$$

then $(D - \alpha)^{n+1}$ annihilates y.

ANNIHILATOR FOR sin βx AND cos βx

If

$$y = e^{\alpha x}(\cos \beta x)(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$$

or

$$y = e^{\alpha x}(\sin \beta x)(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$$

then an annihilator of y is

$$[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^{n+1}$$

THEOREM

Theorem: If L_1 annihilates y_1 and L_2 annihilates y_2, then $L_1 L_2$ annihilates $y_1 + y_2$.

NOTE: This result generalizes for more than two functions added together.

COMMENT

The differential operator that annihilates a function is *not* unique. For example, $D - 5$ annihilates e^{5x}, but so do differential operators of higher order like $D(D - 5)$. When we want a differential annihilator for a function $y = f(x)$, we want the one of lowest possible order that does the job.