Section 4.4

Undetermined Coefficients—Superposition Approach

SOLUTION OF NONHOMOGENEOUS EQUATIONS

Theorem: Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous linear n^{th}-order differential equation on an interval I, and let y_p be any solution of the nonhomogeneous equation on the same interval. Then

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x) + y_p(x)$$

is also a solution of the nonhomogeneous equation on the interval for any constants $c_1, c_2, \ldots, c_k$.

EXISTENCE OF CONSTANTS

Theorem: Let y_p be a given solution of the nonhomogeneous n^{th}-order linear differential equation on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous equation on the interval. Then for any solution $Y(x)$ of the nonhomogeneous equation on I, constants $C_1, C_2, \ldots, C_n$ can be found so that

$$Y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) + y_p(x)$$

GENERAL SOLUTION—NONHOMOGENEOUS EQUATION

Definition: Let y_p be a given solution of the nonhomogeneous linear n^{th}-order differential equation on an interval I, and let

$$y_c = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

denote the general solution of the associated homogeneous equation on the interval. The **general solution** of the nonhomogeneous equation on the interval is defined to be

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x) \\ &= y_c(x) + y_p(x) \end{aligned}$$

COMPLEMENTARY FUNCTION

In the definition from the previous slide, the linear combination

$$y_c = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

is called the **complementary function** for the nonhomogeneous equation. Note that the general solution of a nonhomogeneous equation is

$y =$ *complementary function + any particular solution.*

SUPERPOSITION PRINCIPLE—NONHOMOGENEOUS EQUATION

Theorem: Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear n^{th}-order differential equation on an open interval I corresponding, in turn, to k distinct functions $g_1, g_2, \ldots, g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding DE

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x)$$

Where $i = 1, 2, \ldots, k$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

FINDING A SOLUTION TO A NONHOMOGENEOUS LINEAR DE

To find the solution to a nonhomogeneous linear differential equation with constant coefficients requires two things:

(i) Find the complementary function y_c.

(ii) Find *any* particular solution y_p of the nonhomoegeneous equation.

LIMITATIONS OF THE METHOD OF UNDETERMINED COEFFICIENTS

The **method of undetermined coefficients** is limited to nonhomogeneous equations which

- coefficients are constant and
- $g(x)$ is a constant k, a polynomial function, an exponential function $e^{\alpha x}$, $\sin \beta x$, $\cos \beta x$, or finite sums and products of these functions.

THE PARTICULAR SOLUTION OF *ay″ + by′ +cy = g(x)*

$g(x)$	$y_{p(x)}$
$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$	$x^s(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0)$
$P_n(x)e^{ax}$	$x^s(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0)e^{ax}$
$P_n(x)e^{\alpha x}\sin\beta x$ **or** $P_n(x)e^{\alpha x}\cos\beta x$	$x^s[(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0)e^{\alpha x}\cos\beta x + (A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0)e^{\alpha x}\sin\beta x]$

NOTE: s is the smallest nonnegative integer which will insure that no term of $y_p(x)$ is a solution of the corresponding homogeneous equation.