Section 4.3

Homogeneous Linear Equations with Constant Coefficients

SOLUTIONS TO HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

All solutions of the differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

are, or are constructed from, exponential functions of the form

$$y = e^{mx}.$$

NOTE: $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants.

THE CHARACTERISTIC EQUATION

Substituting $y = e^{mx}$ results in the **auxiliary equation** or **characteristic equation**:

$$am^2 + bm + c = 0.$$

SOLUTIONS TO THE HOMOGENEOUS LINEAR 2ND-ORDER DE

The solutions to the homogeneous linear second-order differential equation depend on the solutions to the characteristic equation. The solutions to the characteristic equation fall into three cases:

- Distinct (unequal) real roots
- Repeated (equal) real roots
- Conjugate complex roots

DISTINCT REAL ROOTS

If the characteristic equation has two distinct (unequal) real roots m_1 and m_2, the two solutions to the DE are $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$. This results in the general solution:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

REPEATED REAL ROOTS

When the roots of the characteristic equation are equal (that is, $m_1 = m_2$), then the two solutions are

$$y_1 = e^{m_1 x} \text{ and } y_2 = x e^{m_1 x}$$

NOTE: The second solution comes from the techniques of Section 4.2.

The general solution is:

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

CONJUGATE COMPLEX ROOTS

If the roots of the characteristic equation are conjugate complex roots

$$m_1 = \alpha + i\beta \text{ and } m_2 = \alpha - i\beta,$$

then the two solutions are

$$y_1 = e^{\alpha x}\cos\beta x \text{ and } y_2 = e^{\alpha x}\sin\beta x,$$

and the general solution is

$$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$