Section 4.1

Initial-Value and Boundary-Value Problems

Linear Dependence and Linear Independence

Solutions of Linear Equations

INITIAL-VALUE PROBLEM

For the n^{th}-order differential equation, the problem

Solve: $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

Subject to: $y(x_0) = y_0, y'(x_0) = y_0', \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)}$,

where y_0, y_0', etc. are arbitrary constants, is called an **initial-value problem**.

EXISTENCE OF A UNIQUE SOLUTION

Theorem: Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$, and $g(x)$ be continuous on an interval and let $a_n(x) \neq 0$ for every x in the interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem exists on the interval and is unique.

BOUNDARY-VALUE PROBLEM

A problem such as

Solve: $a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$

Subject to: $y(a) = y_0,\ y(b) = y_1$,

is called a **boundary-value problem** (BVP). The specified values $y(a) = y_0$ and $y(b) = y_1$ are called **boundary conditions**.

SOLUTION TO BVP

A boundary-variable problem (BVP) can have:

- a unique solution
- several solutions (See Example 6, p. 114.)
- no solution (See Example 7, p. 115.)

LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exists constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2\, f_2(x) + \cdots c_n\, f_n(x) = 0$$

for every x in the interval.

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly independent** on an interval I if it is not linearly dependent on the interval.

THE WRONSKIAN

Theorem: Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ possess at least $n - 1$ derivatives. If the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is not zero for at least one point in the interval I, then the set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is linearly independent on the interval.

NOTE: The determinant above is called the **Wronskian** of the function and is denoted by $W(f_1(x), f_2(x), \ldots, f_n(x))$.

COROLLARY

Corollary: If $f_1(x), f_2(x), \ldots, f_n(x)$ possess at least $n - 1$ derivatives and are linearly dependent on I, then

$$W(f_1(x), f_2(x), \ldots, f_n(x)) = 0$$

for every x in the interval.

HOMOGENEOUS LINEAR EQUATIONS

An linear n^{th}-order differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be **homogeneous**, whereas

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$g(x)$ not identically zero, is said to be **nonhomogeneous**.

SUPERPOSITION PRINCIPLE – HOMOGENEOUS EQUATIONS

Theorem: Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous linear n^{th}-order differential equation on an interval I. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

where the $c_i, i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

TWO COROLLARIES

Corollary 1: A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.

Corollary 2: A homogeneous linear differential equation *always* possesses the trivial solution $y = 0$.

CRITERION FOR LINEARLY INDEPENDENT SOLUTIONS

Theorem: Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear n^{th}-order differential equation on an interval I. Then the set of solutions is **linearly independent** on I if and only if

$$W(y_1, y_2, \ldots, y_n) \neq 0$$

for every x in the interval.

FUNDAMENTAL SET OF SOLUTIONS

Definition: Any linearly independent set $y_1, y_2, \ldots, y_n$ of n solutions of the homogeneous n^{th}-order differential equation on an interval I is said to be a **fundamental set of solutions** on the interval.

EXISTENCE OF CONSTANTS

Theorem: Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear n^{th}-order differential equation on an interval I. Then for any solution $Y(x)$ of the equation on I, constants $C_1, C_2, \ldots, C_n$ can be found so that

$$Y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x).$$

EXISTENCE OF A FUNDAMENTAL SET

Theorem: There exists a fundamental set of solutions for the homogeneous linear n^{th}-order differential equation on an interval I.

GENERAL SOLUTION – HOMOGENEOUS EQUATIONS

Definition: Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear n^{th}-order differential equation on an interval I. The **general solution** (or **complete solution**) of the equation on the interval is defined to be

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, i = 1, 2, \ldots, n$ are arbitrary constants.