Section 2.4

Exact Equations

TOTAL DIFFERENTIAL

Definition: The **total differential** of a function $z = f(x, y)$, with continuous first partial derivatives in a region R in the xy-plane is

$$dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

TOTAL DIFFERENTIAL AND SOLUTION TO A DE

For a family of curves,

$$f(x, y) = c,$$

its total differential $d\, f(x, y) = 0$, and hence is a solution of the first-order differential equation:

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0$$

EXACT DIFFERENTIAL EQUATION

Definition: A differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **exact** if there is a function, $f(x, y)$, whose total differential,

$$d\, f(x, y) = M(x, y)dx + N(x, y)dy$$

CRITERION FOR AN EXACT DIFFERENTIAL EQUATION

Theorem: Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, c < y < d$. The differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

SOLVING AN EXACT EQUATION

1. Show $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
2. Assume $\frac{\partial f}{\partial x} = M(x, y)$. Find f by integrating $M(x, y)$ with respect to x, while holding y constant. We can write

   $$f(x, y) = \int M(x, y)dx + g(y) \quad (1)$$

   where the arbitrary function $g(y)$ is the "constant" of integration.
3. Differentiate (1) with respect to y and set equal to $N(x, y)$. This yields, after solving for $g'(y)$,

   $$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)dx \quad (2)$$

4. Integrate (2) with respect to y and substitute the result into (1). The solution to the DE is $f(x, y) = c$.

INTEGRATING FACTORS

It is sometimes possible to convert a nonexact differential equation into an exact equation by multiplying it by a function $\mu(x, y)$ called an **integrating factor**.

NOTE: It is possible for a solution to be lost or gained as a result of multiplying by an integrating factor.