Section 11.4: The Comparison Tests

The Comparison Test:

The Comparison Test (CT):

(i) If $0 \le a_n \le b_n$ for all $n > N$ and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(iI) If $a_n \ge b_n \ge 0$ for all $n > N$ and if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

NOTES:

1. To be used for positive term series.
2. Any series that is term by term *smaller* than a series *known to converge* must also converge; any series that is term by term *larger* than a series *known to diverge* must also diverge. However, comparing a series to a “larger” series known to diverge or comparing a series to a “smaller” series known to converge tells nothing. *Be careful of the inequality signs*!
3. For comparison, pick a series “close” to the series in question (pick geometric, harmonic, or p-series).

Useful Facts for the Comparison Test:

1. $\ln n < n$
2. $\ln n > 1$ for $n > e$
3. $-1 \le \sin n \le 1$

   $-1 \le \cos n \le 1$

Examples: Use the Comparison Test (CT) to determine if the following series diverge or converge.

1. $$\sum_{n=1}^{\infty} \frac{7}{n^4+3}$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n\sqrt{n}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{2\sqrt{n}}$

<u>The Limit Comparison Test</u>:

The Limit Comparison Test (LCT): Suppose $a_n \geq 0$, $b_n \geq 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

If $0 < L < \infty$, then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either converge or diverge

<u>NOTES</u>:

1. For positive term series.
2. This test is very useful in comparing a very "messy" algebraic series to a "simple" known series. (A term by term comparison in order to use the Comparison Test can be time consuming.)
3. The choice of "simple" known series takes some intuition. For "messy" algebraic series, disregard all but the highest power of n in the numerator and the highest power of n in the denominator and compare to a known p-series. For number raised to "n", the series is either a geometric series or can be compared to a known geometric series.

Examples: Use the Limit Comparison Test (LCT) to determine if the following series converge or diverge.

1. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n-2)^2}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{3n^2+1}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4 \cdot 3^n - 5}$