# <u>Section 11.3</u>: The Integral Test and Estimates of Sums

**<u>The Integral Test</u>:**

**Theorem (The Integral Test):** Let $f$ be a continuous, positive, and decreasing function on the interval $[c, \infty)$, where $c$ is a positive integer, and suppose that $a_n = f(n)$ for all positive integers $n \geq c$. Then $\sum_{n=c}^{\infty} a_n$ converges if and only if the improper integral $\int_c^{\infty} f(x)dx$ converges. In other words:

(i) If $\int_c^{\infty} f(x)dx$ is convergent, then $\sum_{n=c}^{\infty} a_n$ is convergent.

(ii) If $\int_c^{\infty} f(x)dx$ is divergent, then $\sum_{n=c}^{\infty} a_n$ is divergent.

<u>NOTES</u>:

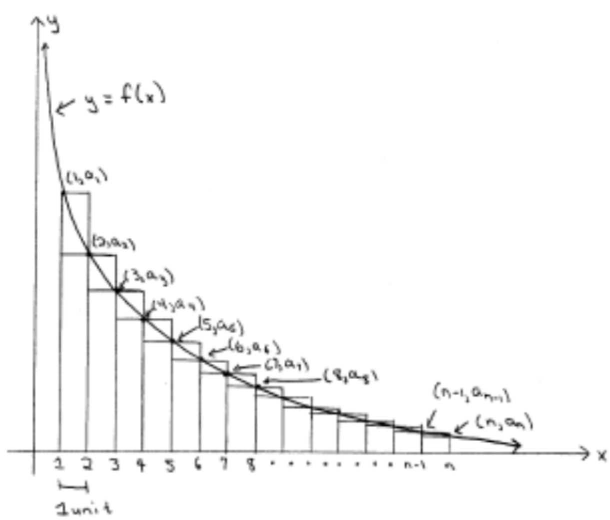
1. The Integral Test is for <u>*positive series*</u> only; that is, when the series has all positive terms.
2. In the application of this test, neither the series nor the integral has to start with 1.
3. This test <u>*does not*</u> give the sum of the series. This test states that the convergence of one implies the convergence of the other. (And the divergence of one implies the divergence of the other.)

<u>Examples</u>: Use the Integral Test to determine if the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

2. $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n}$

**<u>A Proof of the Integral Test</u>:**




The sum $\sum_{k=2}^{n} a_k$ gives the area of the lower rectangles while the sum $\sum_{k=1}^{n-1} a_k$ gives the area of the upper rectangles.

1. Observe that the area under the curve, $A = \int_1^n f(x)dx$, approximates $s_n$ such that

$$\sum_{k=2}^{n} a_k \leq \int_1^n f(x)dx \leq \sum_{k=1}^{n-1} a_k$$

2. Now, observe that, by definition, $s_n = \sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k$, and from 1 we have

$$a_1 + \sum_{k=2}^{n} a_k \leq a_1 + \int_1^n f(x)dx$$

$$s_n \leq a_1 + \int_1^n f(x)dx$$

3. Since $\sum_{k=1}^{n-1} a_k + a_n = \sum_{k=1}^{n} a_k$, we see that $\sum_{k=1}^{n-1} a_k \leq \sum_{k=1}^{n} a_k$. And since $s_n = \sum_{k=1}^{n} a_k$, we have $\sum_{k=1}^{n-1} a_k \leq s_n$. Finally, from 1, we see that $\int_1^n f(x)dx \leq \sum_{k=1}^{n-1} a_n \leq s_n$. Hence, $\int_1^n f(x)dx \leq s_n$. Therefore, from 2 and 3, we have

$$\int_1^n f(x)dx \leq s_n \leq a_1 + \int_1^n f(x)dx$$

and

$$\int_1^\infty f(x)dx \leq \lim_{n\to\infty} s_n \leq a_1 + \int_1^\infty f(x)dx$$

4. Thus, we see that if $\int_1^\infty f(x)dx$ diverges, then $\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} a_k$ also diverges.

5. Finally, we see that if $\int_1^\infty f(x)dx$ converges, then since $a_1$ is finite, $\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} a_k$ also converges. If $\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} a_k$ converges, then $\int_1^\infty f(x)dx$ also converges. ■

**Approximating the Sum of a Series:**

We can use the Integral Test to bound the error on the approximation of a series by summing its first $N$ terms. (Note: The terms of this series must be decreasing.) Consider the following series:

$$\sum_{n=c}^{\infty} a_n = \sum_{n=c}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

The last summation $\sum_{n=N+1}^{\infty} a_n$ is called the ***remainder*** (or ***error***) between the value of the partial sum $s_N = \sum_{n=c}^{N} a_n$ and the true sum of the series.

Observe that remainder is the lower sum of rectangles that we used to define the definite integral. Thus,

$$\sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} f(x)dx$$

and

$$\sum_{n=c}^{\infty} a_n \leq \sum_{n=c}^{N} a_n + \int_N^{\infty} f(x)dx$$

Notice that what this is really saying is that the tail of the series is bounded above by the improper integral.

Example: Find the sum of the following series with an error of less than 0.0001.

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

**$p$-Series**:

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots,$$

where $p$ is a constant is called a ***p-series***.

We will show that a $p$-series

(a) converges if $p > 1$ and
(b) diverges if $p \leq 1$.

NOTE: When $p = 1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *harmonic series*.

To show the two statements in (a) and (b) above, we will use the Integral Test on $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Part 1: For $p \neq 1$.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$
$$= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$
$$= \lim_{t\to\infty} \left[\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

Case 1: If $1 - p > 0$ (*i.e.*, $p < 1$), then $\lim_{t\to\infty} \frac{t^{1-p}}{1-p} = \infty$. So, the improper integral diverges, and by the Integral Test the series diverges if $p < 1$.

Case 2: If $1 - p < 0$ (*i.e.*, $p > 1$), then $\lim_{t\to\infty} \frac{t^{1-p}}{1-p} = 0$. Thus, the improper integral converges to $-\frac{1}{1-p} = \frac{1}{p-1}$. By the Integral Test the series converges.

Part 2: For $p = 1$.

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx \\ &= \lim_{t\to\infty}[\ln|x|]_1^t \\ &= \lim_{t\to\infty}(\ln t - \ln 1) \\ &= \lim_{t\to\infty}(\ln t) = \infty\end{aligned}$$

So, the improper integral diverges, and by the Integral Test, the series diverges.

Therefore, we have shown that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Examples: Determine whether the following series diverge or converge.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

3. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\left(\sqrt[3]{n}\right)}$