Section 1.2

Some Mathematical Models

TERMINOLOGY

A **model** starts by (i) identifying the variables that are responsible for changing the system and (ii) a set of reasonable assumptions about the system. The mathematical construct of all these assumptions is called a **mathematical model** and is often a differential equation or system of differential equations.

MODEL OF A FREELY FALLING BODY

$$\frac{d^2s}{dt^2} = -g$$
$$v(0) = v_0, s(0) = s_0$$

VIBRATION OF A MASS ON A SPRING

Newton's Second Law of Motion: The net force, F, acting on a system in motion is $F = ma$, where m is the mass and a is the acceleration.

Hooke's Law: The restoring force of a stretched spring is opposite to the direction of elongation and is proportional to the amount of elongation. That is, $-k(s + x)$.

VIBRATION OF A MASS ON A SPRING (CONTINUED)

Since the net force, F, on a spring is the resultant of the weight and the restoring force, we get

$$mg - k(s + x) = m\frac{d^2x}{dt^2}$$

In Chapter 5, we will show the net force acting on the mass is $F = -kx$. Thus, we get

$$m\frac{d^2x}{dt^2} = -kx$$
$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \text{ where } \omega^2 = \frac{k}{m}$$

SIMPLE PENDULUM

A **simple pendulum** consists of a rod to which a mass is attached at one end.

For a simple pendulum of length l, at an angle of θ with the vertical, Newton's Second Law gives

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

for small values of θ.

NEWTON'S LAW OF COOLING

The rate at which a body cools is proportional to the difference between the body's temperature and that of the surrounding medium:

$$\frac{dT}{dt} = k(T - T_m), k < 0$$

POPULATION GROWTH

The rate at which a population expands is proportional to the population present at that time,

$$\frac{dP}{dt} = kP, k > 0$$

SPREAD OF A DISEASE

The rate at which a disease spreads is proportional to both the number of people infected, $x(t)$, and the number of people not yet exposed, $y(t)$:

$$\frac{dx(t)}{dt} = k\, x(t)\, y(t)$$

SPREAD OF A DISEASE (CONTINUED)

However, if one infected person enters a town of n people, then $x + y = n + 1$.

Hence,

$$\frac{dx(t)}{dt} = k\, x(t)[n + 1 - x(t)]$$

with $x(0) = 1$.