<u>Section 11.9</u>: Representations of Functions as Power Series

<u>Differentiation and Integration of Power Series:</u>

If the domain of a power series (*i.e.*, the interval of convergence) is not a single point, then (with the possible exception of endpoints—see note below):

1. f is differentiable in the same domain:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

and

$$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$\text{or } f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

NOTE: $f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$

2. and the integral can be determined:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

and

$$\int f(x)dx = C + c_0(x-a) + \tfrac{1}{2}c_1(x-a)^2 + \tfrac{1}{3}c_2(x-a)^3 + \tfrac{1}{4}c_3(x-a)^4 + \cdots$$

$$\text{or } \int f(x)dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

NOTES:

1. This result states that functions defined by power series behave <u>*exactly*</u> like polynomial functions; *i.e.*, they are continuous on their interval of convergence, and derivatives and antiderivatives can be found just like for polynomials (by differentiating and integrating each term)
2. After taking the derivative and if the original series has convergent endpoint(s), then check the endpoints of the new series. There is no need to check endpoints if the original series <u>did not have convergent endpoints</u>.
3. After integration, check endpoints even if the original series has no convergent endpoints.

Example: Suppose

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

(a) Find the interval of convergence of $f(x)$.
(b) Find $f'(x)$ and find its interval of convergence.
(c) Find $\int f(x)dx$, and find its interval of convergence.

(a) We use the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(x-1)^{n+1}}{n+1}}{\frac{(-1)^{n+1}(x-1)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n+1}\cdot\frac{n}{(x-1)^n}\right|$$

$$= |x-1|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= |x-1|\cdot 1$$

$$= |x-1| < 1$$

Thus, the radius of convergence is $\frac{1}{1} = 1$. By solving the absolute value equation, we find that the interval on convergence is at least $0 < x < 2$ or $(0, 2)$.

Now, we must check the endpoints.

***x* = 0**:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(0-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n}{n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}$$

$$= \sum_{n=1}^{\infty}\frac{-1}{n}$$

This series diverges since it is -1 times the harmonic series.

***x* = 2**:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

This series converges since it is the alternating harmonic series. Recall the alternating harmonic series converges by the AST.

So, the interval of convergence for $f(x)$ is $0 < x \leq 2$ or $(0, 2]$.

(b) Now, let's take the derivative of $f(x)$.

$$f'(x) = \frac{d}{dx}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}\right]$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n (x-1)^{n-1}}{n}$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^{n-1}$$

By the result on page 1, we know this series converges at least on the interval $0 < x < 2$. Since the original power series did not converge at the endpoint $x = 0$, the series found by taking the derivative does not converge at $x = 0$. So, we only need to check the endpoint $x = 2$.

***x* = 2**:

$$\sum_{n=1}^{\infty} (-1)^{n+1}(2-1)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1}(1)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1}$$

This series diverges by the n^{th} Term Test.

So, the interval of convergence of $f'(x)$ is $0 < x < 2$ or $(0, 2)$.

(c) Now, let's take the integral of $f(x)$.

$$\int \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}\right] dx = C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n(n+1)}$$

By the result on page 1, we know that this new series converges for $0 < x < 2$ or $(0, 2)$. However, we need to check the endpoints. (When integrating, the new series may become convergent at the endpoints.)

***x* = 0**:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0-1)^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+2}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Now, using the LCT and comparing the series to a p-series with $p = 2$ $\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)$, we find the series converges since

$$\lim_{m \to \infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{1}{n(n+1)} \cdot \frac{n^2}{1}$$
$$= \lim_{n \to \infty} \frac{n^2}{n^2 + n} = 1$$

<u>x = 2</u>:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2-1)^{n+1}}{n(n+1)}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1)^{n+1}}{n(n+1)}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}$$

This series converges by the AST since $\lim_{n\to\infty}\frac{1}{n(n+1)}=0$ and the series in decreasing, that is

$$\frac{1}{(n+1)(n+2)}<\frac{1}{n(n+1)}.$$

Thus the interval of convergence for the antiderivative of the power series is $0\le x\le 2$ or $[0,2]$.

<u>Power Series as Functions</u>:

A power series represents an infinite series <u>and</u> a function with a specific domain. Consider the geometric series

$$\sum_{n=1}^{\infty}x^{n-1}=\sum_{n=0}^{\infty}x^{n}=1+x+x^2+x^3+x^4+\cdots$$

where $a=1$ and $r=x$. The interval of convergence, $|x|<1$ or $-1<x<1$, determines the domain of the function, $(-1,1)$. Since we know a formula for the sum of a convergent geometric series, we can say that

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^{n}\quad\text{for } x \text{ in } (-1,1)\,.$$

<u>Examples</u>:

1. Convert

$$f(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{3^n}$$

to function notation. State the domain.

$$f(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{3^n}=\sum_{n=0}^{\infty}\left(\frac{x^2}{3}\right)^n=\frac{1}{1-\frac{x^2}{3}}=\frac{3}{3-x^2}$$

$$\text{for } \left|\frac{x^2}{3}\right|<1 \text{ or } -\sqrt{3}<x<\sqrt{3}$$

So, $f(x)=\frac{3}{3-x^2}$ for $-\sqrt{3}<x<\sqrt{3}$.

2. Find the geometric series represented by $f(x) = \frac{x}{2x+3}$ centered at $a = 0$.

$$f(x) = \frac{x}{2x+3} = \frac{x}{3}\left(\frac{1}{\frac{2x}{3}+1}\right) = \frac{x}{3}\cdot\frac{1}{1-\left(-\frac{2x}{3}\right)}$$

$$= \frac{x}{3}\sum_{n=0}^{\infty}\left(-\frac{2x}{3}\right)^n = \frac{x}{3}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n x^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n x^{n+1}}{3^{n+1}}$$

The domain: $\left|\frac{2x}{3}\right| < 1$ or $|x| < \frac{3}{2}$ or $-\frac{3}{2} < x < \frac{3}{2}$.

So,

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n x^{n+1}}{3^{n+1}} \quad \text{for } -\frac{3}{2} < x < \frac{3}{2}.$$

3. Develop a power series for $f(x) = \frac{3}{4-x}$ centered at $a = -2$.

$$f(x) = \frac{3}{4-x} = \frac{3}{6-(x+2)} = \frac{3}{6}\cdot\frac{1}{1-\left(\frac{x+2}{6}\right)}$$

So,

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x+2}{6}\right)^n$$

where $\left|\frac{x+2}{6}\right| < 1$ or $|x+2| < 6$.

Hence, the radius of convergence is $r = 6$, and the interval of convergence is $-8 < x < 4$. Thus,

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(x+2)^n}{2^n 3^n} = \sum_{n=0}^{\infty}\frac{(x+2)^n}{2^{n+1}3^n} \quad \text{for } -8 < x < 4\,.$$

Operations on Power Series:

Let $f(x) = \sum c_n x^n$ and $g(x) = \sum d_n x^n$. Then

i. $f(kx) = \sum c_n k^n x^n$

ii. $f(x^N) = \sum c_n x^{nN}$

iii. $f(x) \pm g(x) = \sum (c_n \pm d_n) x^n$

NOTE: These operations *may* change the interval of convergence of the power series.

Example: Determine a power series for the function

$$f(x) = \frac{2x}{x^2 - 1}.$$

By using partial fraction decomposition, we see that

$$f(x) = \frac{2x}{x^2 - 1} = \frac{1}{1+x} - \frac{1}{1-x}.$$

From page 3, we know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n .$$

Substituting $-x$ for x, we see that

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n .$$

Thus,

$$\begin{aligned} \frac{2x}{1-x^2} &= \sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} [(-1)^n - 1] x^n \\ &= \sum_{n=0}^{\infty} (-2) x^{2n+1} \end{aligned}$$

The last line coming from the fact that when n is odd the coefficient of x^n is -2 and when n is even the coefficient is 0. Using the Ratio Test, we can show that the interval of convergence is $-1 < x < 1$. (*Verify this!*) The endpoints are not included. (*Verify this also!*)

Using Differentiation and Integration to Represent Functions as Power Series:

Examples:

1. Find a power series representation for $f(x) = \frac{2}{(x+1)^3}$ centered at $a = 0$.

   We begin by observing that

$$\int f = \int 2(x+1)^{-3} dx = -(x+1)^{-2}$$

and

$$\iint f = \int -(x+1)^{-2} dx = (x+1)^{-1} = \frac{1}{x+1}$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

for $-1 < x < 1$ since geometric series.

Next, we take the derivative of the above series twice.

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \sum_{n=1}^{\infty}(-1)^n n x^{n-1} = \sum_{n=0}^{\infty}(-1)^n n x^{n-1}$$

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n n x^{n-1}\right] = \sum_{n=1}^{\infty}(-1)^n n(n-1)x^{n-2}$$

$$= \sum_{n=0}^{\infty}(-1)^n n(n-1)x^{n-2}$$

The last series converges for $-1 < x < 1$.

NOTE: There is no need to check the endpoints since before taking derivatives of the series, the geometric series did not have convergent endpoints.

2. Find a series representation for $f(x) = \arctan x$.

We first note that up to a constant

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx$$

Now, we know (by geometric series) that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

for $-1 < x < 1$. Thus,

$$\frac{1}{1+x^2} = 1 - x^2 + (x^2)^2 - (x^2)^3 + (x^2)^4 - \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

The above series converges for $|x^2| < 1$ or $-1 < x < 1$.

Now, we integrate the above power series to get

$$\tan^{-1} x = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

for $-1 < x < 1$. Substituting $x = 0$, we find that $C = 0$. Hence,

$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

which converges for $-1 < x < 1$. However, since we integrated a power series, we must check to see if the new series converges at the endpoints.

Check $x = -1$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+1}$$

which converges by the Alternating Series Test since

$$\lim_{n\to\infty} \frac{1}{2n+1} = 0 \quad \text{and} \quad \frac{1}{2n+3} < \frac{1}{2n+1}.$$

Check $x = 1$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

which converges by the Alternating Series Test (see above).
Consequently, we find that

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for} \quad -1 \le x \le 1.$$

Exercises

1. Using a geometric series, develop a power series centered at a for the following functions. State the domain. (Be Sure to check endpoint if applicable.)

(a) $f(x) = \dfrac{2}{5-x}$ about $a = 2$ **Ans**: $\displaystyle\sum_{n=0}^{\infty} \frac{2(x-2)^n}{3^{n+1}}, 1 < x < 5$

(b) $f(x) = \dfrac{1}{(1-x^2)^2}$ about $a = 0$ **Ans**: $\displaystyle\sum_{n=0}^{\infty} nx^{2n-2}, -1 < x < 1$

(c) $f(x) = \ln(x^2 + 1)$ about $a = 0$ **Ans**: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}, -1 \le x \le 1$

(d) $f(x) = \dfrac{x+1}{x^3+1}$ about $a = 0$ **Ans**: $\displaystyle\sum_{n=0}^{\infty} (x+1)(-1)^n x^{3n}, -1 \le x < 1$

HINT: $\dfrac{x+1}{x^3+1} = \dfrac{x}{x^3+1} + \dfrac{1}{x^3+1}$

2. Convert the following power series to functional notation. State the domain. (Be sure to check endpoints if applicable.)

(a) $\displaystyle\sum_{n=0}^{\infty} x^{n+1}$ **Ans**: $f(x) = \dfrac{x}{1-x}, -1 < x < 1$

(b) $\displaystyle\sum_{n=1}^{\infty} -n(-x)^{n-1}$ **Ans**: $f(x) = -\dfrac{1}{(1+x)^2}, -1 < x < 1$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3^n} x^{2n-1}$ **Ans**: $f(x) = \dfrac{6x}{(3-x^2)^2}, -\sqrt{3} < x < \sqrt{3}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ **Ans**: $f(x) = -\ln|1-x|, -1 \le x < 1$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ **Ans**: $f(x) = \arctan x = \tan^{-1} x, -1 \le x \le 1$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (1-x)^{n+1}}{n+1}$ **Ans**: $f(x) = \ln|2-x|, 0 \le x < 2$