<u>Section 11.8</u>: Power Series

<u>Introduction</u>:

So far we have only talked about series of constants. For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}.$$

We asked if such series converge or diverge.
Now, we want to discuss series of *functions*. For example,

$$\sum_{n=1}^{\infty} \frac{\sin nx}{2^n}.$$

Now the question we ask is:

For what values of x does the series converge?

Another related question is:

If the series converges for some values of x, what function does it converge to; that is, what is $s(x)$?

We are only going to discuss very special series of functions called *power series*.

<u>Power Series and Convergence</u>:

A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a *<u>power series</u>* centered at $x = a$. Each partial sum is a polynomial.

<u>Example</u>:

$$\sum_{n=0}^{\infty} \frac{(2x+1)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n\left(x+\frac{1}{2}\right)^n}{n!}$$

$$= 1 + 2\left(x+\tfrac{1}{2}\right) + \frac{2^2\left(x+\frac{1}{2}\right)^2}{2} + \frac{2^3\left(x+\frac{1}{2}\right)^3}{3!} + \frac{2^4\left(x+\frac{1}{2}\right)^4}{4!} + \cdots$$

The *<u>coefficients</u>* are: $c_0 = 1,\ c_1 = 2,\ c_2 = 2,\ c_3 = \frac{4}{3},\ c_4 = \frac{2}{3}, \ldots$.
The *<u>center</u>* is: $a = -\frac{1}{2}$.

The infinite power series is a function of x defined for those values of x for which the series converges. The set of values of x for which the power series converges is called the *<u>interval of convergence</u>* or the *<u>convergence set</u>*.

The interval of convergence, $a - r < x < a + r$, has a *<u>radius of convergence</u>*, r.

For the power series,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

we use the Ratio Test to determine the domain of $f(x)$ and thus determine the convergence set.

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)}{c_n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$$

Notice the Ratio Test tells us that

$$|x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$$

must be less than 1 for the series to converge. Let

$$\rho = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$$

and consider the following three cases:

CASE 1 ($\boldsymbol{\rho = 0}$): If $\rho = 0$, then the power series is convergence for all x since

$$|x-a|\rho = 0 < 1\,.$$

The interval of convergence is $-\infty < x < \infty$ or $(-\infty, \infty)$ and the radius of convergence is $r = \infty$.

Example:

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0 \text{ for all values of } x$$

center: $a = 0$
radius of convergence: $r = \infty$
convergence set: $(-\infty, \infty)$

CASE 2 ($\rho = \infty$): If $\rho = \infty$, then the power series converges for $x = a$ only since by the Ratio Test the series diverges for all values of x except $x = a$ ($r = 0, x = a$).

Example:

$$\sum_{n=0}^{\infty} n!\,(x-2)^n$$

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,(x-2)^{n+1}}{n!\,(x-2)^n}\right| = |x-2|\lim_{n\to\infty}(n+1) = \infty$$

for all values of x except $a = 2$

center: $a = 2$
radius of convergence: $r = 0$
convergence set: $x = 2$

CASE 3 ($\rho \neq 0$ and $\rho \neq \infty$): If $\rho \neq 0$ and $\rho \neq \infty$, then by the Ratio Test,

$$|x-a|\,\rho$$

must be less than 1 for the series to converge. The series, not counting endpoints, *converges absolutely* for those values of x such that

$$|x-a|\,\rho < 1 \ \text{ or } \ |x-a| < \frac{1}{\rho}.$$

Thus, $\frac{1}{\rho}$ is the radius of convergence.

To determine whether the endpoints are included in the domain (interval of convergence), a test *other than the Ratio Test* must be used. (Recall when $|x-a|\,\rho = 1$ the Ratio Test fails.)

Example:

$$\sum_{n=1}^{\infty}\frac{n(2x+1)^n}{n+1} = \sum_{n=1}^{\infty}\frac{n\cdot 2^n\left(x+\frac{1}{2}\right)^n}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{\dfrac{(n+1)\cdot 2^{n+1}\left(x+\frac{1}{2}\right)^{n+1}}{n+2}}{\dfrac{n\cdot 2^n\left(x+\frac{1}{2}\right)^n}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot 2^{n+1}\left(x+\frac{1}{2}\right)^{n+1}}{n+2}\cdot\frac{n+1}{n\cdot 2^n\left(x+\frac{1}{2}\right)^n}\right|$$

$$= \left|x+\tfrac{1}{2}\right|\lim_{n\to\infty}\frac{2(n+1)^2}{n(n+2)}$$

$$= \left|x+\tfrac{1}{2}\right|\lim_{n\to\infty}\frac{2n^2+4n+2}{n^2+2n}$$

$$= \left|x+\tfrac{1}{2}\right|2$$

Thus, the series converges when

$$\left|x+\tfrac{1}{2}\right|2<1 \quad \text{or} \quad \left|x+\tfrac{1}{2}\right|<\frac{1}{2}$$

Hence,

$$-\frac{1}{2}<x+\frac{1}{2}<\frac{1}{2}$$
$$-\frac{1}{2}-\frac{1}{2}<x<\frac{1}{2}-\frac{1}{2}$$
$$-1<x<0$$

center: $a=-\frac{1}{2}$

radius of convergence: $r=\frac{1}{2}$

Now, before stating the interval of convergence, we need to check the endpoints of the interval, namely $x=-1$ and $x=0$.

When $x=-1$, the power series is

$$\sum_{n=1}^{\infty}\frac{n(-1)^n}{n+1}.$$

This series diverges by the n^{th} Term Test since

$$\lim_{n\to\infty}\frac{n}{n+1}=1.$$

When $x=0$, the power series is

$$\sum_{n=1}^{\infty}\frac{n}{n+1}$$

which also diverges by the n^{th} Term Test since

$$\lim_{n\to\infty}\frac{n}{n+1}=1.$$

Thus, the interval of convergence is

$$-1<x<0 \;\text{ or }\; (-1,0)$$

the radius of convergence is

$$r=\frac{1}{2},$$

and the center is

$$a=-\frac{1}{2}.$$

<u>Additional Examples</u>: Find the interval and radius of convergence of the following power series.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^{2n}}{n^3}$