<u>Section 11.6</u>: Absolute Convergence and the Ratio and Root Tests

<u>Absolute and Conditional Convergence</u>:

Absolute Convergence Test (ACT): If $\sum|a_n|$ converges, then $\sum a_n$ also converges.

Definitions:

1. $\sum a_n$ is ***<u>absolutely convergent</u>*** or ***<u>converges absolutely</u>*** (CA) if $\sum|a_n|$ converges.
2. $\sum a_n$ is ***<u>conditionally convergent</u>*** or ***<u>converges conditionally</u>*** (CC) if $\sum a_n$ converges but $\sum|a_n|$ diverges.

<u>NOTE</u>: After determining convergence by the Alternating Series Test (AST), then use Integral Test, Comparison Test (CT), or Limit Comparison Test (LCT) on $\sum|a_n|$ to determine absolute convergence (CA) or conditional convergence (CC).

<u>Examples</u>:

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$

3. $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n-1} \ln n}{n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^{3/2}}$

<u>The Ratio Test</u>:

The Ratio Test (RT): Let $\sum a_n$ be a series of nonzero terms and suppose

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho\,.$$

(i) If $\rho < 1$, the series converges absolutely.
(ii) If $\rho > 1$, the series diverges.
(iii) If $\rho = 1$, the test is inclusive.

<u>NOTES</u>:

1. For any type of series: positive, alternating, or other.
2. If $\rho = 1$, the test fails. You *<u>must</u>* use a different test.
3. If $\rho = +\infty$, the series diverges (ρ does not have to be finite for this test).
4. If $\rho = 0$, the series converges (ρ can have a value of 0 in this test).
5. This test is most useful with series involving powers and factorials.

<u>Useful Facts for Factorials</u>:

1. $2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n) = 2^n n!$

2. $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1) = \dfrac{(2n+1)!}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} = \dfrac{(2n+1)!}{2^n n!}$

<u>Examples</u>:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n!}$

2. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)}{4^n}$$

<u>The Root Test</u>:

The Root Test (RoT): Let $\sum a_n$ be a series of nonzero terms and suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho \,.$$

(i) If $\rho < 1$, the series converges absolutely.
(ii) If $\rho > 1$, the series diverges.
(iii) If $\rho = 1$, the test is inclusive.

<u>NOTES</u>:

1. For any type of series: positive, alternating, or other.
2. If $\rho = 1$, the test fails. You <u>*must*</u> use a different test.
3. If $\rho = +\infty$, the series diverges (ρ does not have to be finite for this test).
4. If $\rho = 0$, the series converges (ρ can have a value of 0 in this test).
5. This test is most useful for series involving powers only.
6. For series involving both powers and factorials use the Ratio Test (RT).

Example: Use the Root Test to determine if the following series diverges or converges.

$$\sum_{n=1}^{\infty}\left(\frac{n+1}{2n+1}\right)^n$$

Section 11.7: Strategy for Testing Series

1. If $\lim_{n\to\infty} a_n \neq 0$, conclude from the n^{th} Term Test that the series diverges.
2. If a_n involves $n!$, or r^n, try the Ratio Test.
3. If a_n involves n^n, try the Root Test (or possibly the Ratio Test).
4. If the series is alternating, then obviously try the Alternating Series Test. (Don't forget to determine absolute or conditional convergence.)
5. If a_n is a positive series and involves only constant powers of n, try the Limit Comparison Test. In particular, if a_n is a rational expression in n, use this test with b_n as the quotient of the leading terms from numerator and denominator.
6. If the tests above do not work and the series is positive, try the Comparison Test or the Integral Test.
7. If all else fails, try some clever manipulation or a neat "trick" to determine convergence or divergence.

Summary of Tests for Convergence/Divergence of Series

Test	**Convergence**	**Divergence**
n^{th}-Term Test	$\lim_{n\to\infty} a_n = 0$, may converge or diverge. Use another test.	$\lim_{n\to\infty} a_n \neq 0$
Geometric Series	$\|r\| < 1$	$\|r\| \geq 1$
p-Series	$p > 1$	$p \leq 1$
Integral Test (The function f must be decreasing to apply this test.)	See improper integrals.	
Ratio/Root Test (Inconclusive if $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$.)	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$, absolute convergence	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$
Limit Comparison Test (The convergence or divergence of $\sum b_n$ is known.)	If $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ (but not infinite), then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.	
Comparison Test (The convergence or divergence of $\sum b_n$ is known.)	$\sum a_n$ converges if $\sum b_n$ converges and $a_n \leq b_n$.	$\sum a_n$ diverges if $\sum b_n$ diverges and $a_n \geq b_n$.
Alternating Series Test	i. $a_{n+1} \leq a_n$ ($\{a_n\}$is decreasing) and ii. $\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$
Improper Integrals	The limit is a finite number.	The limit either does not exist or is infinite.