# <u>Section 11.10</u>: Taylor and Maclaurin Series

**<u>The Uniqueness Theorem</u>:** Suppose $f$ satisfies

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

for all $x$ in some interval around $a$. Then

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Thus, a function cannot have more than one power series in $x - a$ that represents it.

<u>NOTES</u>:

1. Consider $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and the following derivatives:

$$\begin{aligned}
f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \\
f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \\
f''(x) &= 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \cdots \\
f'''(x) &= 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + \cdots \\
f^{(4)}(x) &= 4\cdot 3\cdot 2c_4 + \cdots
\end{aligned}$$

In each case, if $x = a$, we have

$$\begin{aligned}
f(a) &= c_0 = 0!\,c_0 \\
f'(a) &= c_1 = 1!\,c_1 \\
f''(a) &= 2!\,c_2 \\
f'''(a) &= 3!\,c_3 \\
f^{(4)}(a) &= 4!\,c_4 \\
&\vdots \\
f^{(n)}(a) &= n!\,c_n
\end{aligned}$$

Solving for each coefficient:

$$c_0 = \frac{f(a)}{0!};\quad c_1 = \frac{f'(a)}{1!};$$
$$c_2 = \frac{f''(a)}{2!};\quad c_3 = \frac{f'''(a)}{3!};$$
$$c_4 = \frac{f^{(4)}(a)}{4!};\quad \text{and}$$
$$c_n = \frac{f^{(n)}(a)}{n!}$$

2. A power series

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

where coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

is called a <u>*Taylor series*</u>.

3. When $a = 0$, the series in known as a <u>*Maclaurin series*</u>.
4. The last part of the Uniqueness Theorem tells us that there is <u>only one</u> power series representation for a function. In particular, the geometric series and differentiation/integration techniques of the last section yield Taylor and Maclaurin series.

Examples:

1. Find the Maclaurin series for $f(x) = \sin x$.

$$\begin{aligned} f(x) &= \sin x & f(0) &= \sin 0 = 0 \\ f'(x) &= \cos x & f'(0) &= \cos 0 = 1 \\ f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0 \\ f'''(x) &= -\cos x & f'''(0) &= -\cos 0 = -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0 \\ &\vdots & &\vdots \end{aligned}$$

So,

$$\begin{aligned} f(x) &= \sin x \\ &= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \cdots \\ &= 0 + x + 0 + \frac{(-1)}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 + \frac{(-1)}{7!}x^7 + \cdots \\ &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

Now, we use the Ratio Test to find the domain.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{[2(n+1)+1]!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| &= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^2 \cdot x^{2n+1}}{(2n+3)(2n+2)(2n+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| \\ &= |x^2| \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} \end{aligned}$$

Note that

$$\lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0\,.$$

Thus, $\rho = 0$ which implies that the radius of convergence is $r = \infty$ Hence, the domain is all real numbers. So,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for } -\infty < x < \infty\,.$$

2. Find the Maclaurin series for $f(x) = \cos x$.

Now, we could use the same process as we did in Example 1. However, it is easier if we recognize that $\cos x = \frac{d}{dx}(\sin x)$. Thus,

$$\begin{aligned}\cos x &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\end{aligned}$$

Since the domain for $\sin x$ is all real numbers, the domain for $\cos x$ is also all real numbers; *i.e.*, $-\infty < x < \infty$ or $(-\infty, \infty)$.

3. Find the Maclaurin series for $g(x) = e^{x^2}$.

Let $f(x) = e^x$ so that $g(x) = f(x^2)$.

$$\begin{aligned} f(x) &= e^x & f(0) &= e^0 = 1 \\ f'(x) &= e^x & f'(0) &= e^0 = 1 \\ f''(x) &= e^x & f''(0) &= e^0 = 1 \\ &\vdots & &\vdots \end{aligned}$$

Thus,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and

$$\begin{aligned} e^{x^2} &= g(x) = f(x^2) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}. \end{aligned}$$

So,

$$\left.\begin{aligned} f(x) = e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x \\ &\text{and} \\ g(x) = e^{x^2} &= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \quad \text{for all } x \end{aligned}\right\} \text{Verify domain using Ratio Test!}$$

4. Find the Taylor series for $f(x) = \ln x$ centered at $x = 1$.

Method 1: (Geometric Series)

$$f(x) = \ln x$$
$$f'(x) = \frac{1}{x} = \frac{1}{1-(1-x)} = \frac{1}{1+(x-1)}$$
$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad \text{for } |x-1| < 1 \quad \text{or} \quad 0 < x < 2$$

So,

$$\int \left[ \sum_{n=0}^{\infty} (-1)^n (x-1)^n \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}.$$

Hence, $\ln x = C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$ . Substituting 1 for $x$, we find that $C = 0$.

Thus,

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} \quad \text{for} \quad 0 < x \leq 2.$$

Verify the convergence and divergence at the endpoints.

Method 2: (Taylor's Theorem)

$$\begin{aligned} f(x) &= \ln x & f(1) &= \ln 1 = 0 \\ f'(x) &= x^{-1} & f'(1) &= 1^{-1} = 1 \\ f''(x) &= -x^{-2} & f''(1) &= -(1)^{-2} = -1 \\ f'''(x) &= 2x^{-3} & f'''(1) &= 2 \cdot 1^{-3} = 2 \\ f^{(4)}(x) &= -6x^{-4} & f^{(4)}(1) &= -6 \cdot 1^{-4} = -6 \\ &\vdots & &\vdots \\ f^{(n)}(x) &= (-1)^{n-1}(n-1)!\, x^{-n} & f^{(n)}(1) &= (-1)^n (n-1)! \cdot 1^{-n} = (-1)^{n-1}(n-1)! \\ &\vdots & &\vdots \end{aligned}$$

Hence,

$$\begin{aligned} f(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \cdots \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4 + \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n + \cdots \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n + \cdots \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} \quad \text{for} \quad 0 < x \leq 2 \end{aligned}$$

Verify the domain by using the Ratio Test.

5. Find the Maclaurin series for

$$f(x) = \frac{x^3}{4 + 9x^2}.$$

Method 1: (Geometric Series)

$$f(x) = \frac{x^3}{4 + 9x^2} = \frac{x^3}{4} \cdot \frac{1}{1 - \left(-\frac{9}{4}x^2\right)}$$
$$= \frac{x^3}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{9}{4}x^2\right)^n$$

So,

$$f(x) = \frac{x^3}{4} \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n}}{2^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n+3}}{2^{2n+2}}.$$

Since this series came from a geometric series, the domain (interval of convergence) is:

$$\left|\frac{9}{4}x^2\right| < 1$$
or
$$|x^2| < \frac{4}{9}$$
or
$$|x| < \frac{2}{3}$$
or
$$-\frac{2}{3} < x < \frac{2}{3}$$

Method 2: (Taylor's Theorem)

Let $g(x) = \frac{1}{1+x}$. Then, $f(x) = \frac{x^3}{4} \cdot g\left(\frac{9x^2}{4}\right)$.

$$\begin{aligned} g(x) &= (1+x)^{-1} & g(1) &= (1+0)^{-1} = 1 \\ g'(x) &= -(1+x)^{-2} & g'(1) &= -(1+0)^{-2} = -1 \\ g''(x) &= 2(1+x)^{-3} & g''(1) &= 2(1+0)^{-3} = 2 \\ g'''(x) &= -6(1+x)^{-4} & g'''(1) &= -6(1+0)^{-4} = -6 \\ &\vdots & &\vdots \\ g^{(n)}(x) &= (-1)^n n!\,(1+x)^{-n-1} & g^{(n)}(1) &= (-1)^n n!\,(1+0)^{-n} - 1 = (-1)^n n! \\ &\vdots & &\vdots \end{aligned}$$

So,

$$\begin{aligned} g(x) &= g(0) + \frac{g'(0)}{1!}(x-0) + \frac{g''(0)}{2!}(x-0)^2 + \frac{g'''(0)}{3!}(x-0)^3 + \frac{g^{(4)}(0)}{4!}(x-0)^4 + \cdots \\ &= 1 - x + \frac{2}{2}x^2 + \frac{-6}{6}x^3 + \frac{24}{24}x^4 + \cdots \\ &= 1 - x + x^2 - x^3 + x^4 - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \end{aligned}$$

Thus,

$$f(x) = \frac{x^3}{4} \cdot g\left(\frac{9x^2}{4}\right)$$
$$= \frac{x^3}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{9x^2}{4}\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n+3}}{2^{2n+2}} \quad \text{for} \quad -\frac{2}{3} < x < \frac{2}{3}$$

Verify the domain by the Ratio Test!

6. Find the Maclaurin series for $f(x) = (1+x)^k$, where $k$ is any real number.

Using Taylor's Theorem, we have

$$f(x) = (1+x)^k$$
$$f'(x) = k(1+x)^{k-1}$$
$$f''(x) = k(k-1)(1+x)^{k-2}$$
$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$$
$$\vdots$$
$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n}$$

For $x = 0$, we find that

$$f(0) = 1$$
$$f'(0) = k$$
$$f''(0) = k(k-1)$$
$$f'''(0) = k(k-1)(k-2)$$
$$\vdots$$
$$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$$

Therefore, the Maclaurin series for $f(x) = (1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n .$$

This series is called the **binomial series**. Using the Ratio Test, this series will converge if $|x| < 1$ and diverge if $|x| > 1$. (Verify this!) Convergence at the endpoints, $\pm 1$, depends on the value for $k$.

The traditional notation for the coefficients of the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the **binomial coefficients**.

**THE BINOMIAL SERIES:** If $k$ is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$$

The series diverges at the endpoints, $\pm 1$, if $k \leq -1$. The series diverges at the endpoint $-1$ and converges at the endpoint 1 if $-1 < k \leq 0$. The series converges at both endpoints, $\pm 1$, if $k \geq 0$.

7. Represent $f(x) = \sqrt{1+x}$ as a Maclaurin series.

Using the Binomial Series above, we find that

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$$

**Please note the table of important Maclaurin series on page 779 of the text.**

**Exercises:**

1. Find the Maclaurin series for $f(x) = x\cos(3x)$. Use the power series for $\cos x$ that we developed in Example 2 on page 3. State the domain.

2. Find the Maclaurin series for $g(x) = \arctan x$. Use this result to find the Maclaurin series for $f(x) = \frac{\arctan(x^3)}{x}$. State the domain.

3. Find the Taylor series for $f(x) = \ln(2x+3)$ about $a = -1$. State the domain.

4. Find the Maclaurin series for $f(x) = x^2\sin(x^2)$. Use the power series for $\sin x$ that we developed in Example 1 on page 2. State the domain.

**ANSWERS:**

1. $$x\cos(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n+1}}{(2n)!}, \qquad -\infty < x < \infty$$

2. $$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad -1 \le x \le 1$$

$$\frac{\arctan(x^3)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{2n+1}, \qquad -1 \le x \le 1$$

3. $$\ln(2x+3) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}(x+1)^{n+1}}{n+1}, \qquad -\frac{3}{2} \le x \le \frac{3}{2}$$

4. $$x^2\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)!}, \qquad -\infty < x < \infty$$