INTEGRATING FACTORS AND EXACT EQUATIONS

If the differential equation $M(x,y)dx + N(x,y)dy = 0$ is not exact, it may be possible to make it exact by multiplying by an appropriate factor $\mu(x,y)$, which is called an **integrating factor** for the differential equation.

Theorem: Consider the differential equation $M(x,y)dx + N(x,y)dy = 0$.

1. If

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = h(x)$$

is a function of x alone, then $e^{\int h(x)dx}$ is an integrating factor.

2. If

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = k(y)$$

is a function of y alone, then $e^{\int k(y)dy}$ is an integrating factor.

NOTE: If either $h(x)$ or $k(y)$ is constant, this theorem still applies.

Exercises: Find an integrating factor and solve the given equation.

1. $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$

2. $dx + \left(\frac{x}{y} - \sin y\right) dy = 0$

3. $y\,dx + (2xy - e^{2x})dy = 0$

Answers:

1. $\mu(x) = e^{3x}$; $(3x^2y + y^3)e^{3x} = c$

2. $\mu(y) = y$; $xy + y\cos y - \sin y = c$

3. $\mu(y) = \frac{e^{2y}}{y}$; $xe^{2y} - \ln|\,y\,| = c$