COMPLEX NUMBERS

Introduction to Complex Numbers:

1. The number i is the number that is a solution to the equation

$$z^2 + 1 = 0$$

or

$$z^2 = -1$$

or

$$i^2 = -1$$

2. If we enlarge the real numbers to a new set including the number i, we obtain the *complex number system* $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$.

   (a) All familiar operations of addition, subtraction, multiplication, and division can be defined and performed.

   - $(a + ib) + (c + id) = (a + c) + i(b + d)$
   - $(a + ib) - (c + id) = (a - c) + i(b - d)$
   - $(a + ib) \cdot (c + id) = ac + adi + bci + bdi^2 = (ac - bd) + i(ad + bc)$
   - $\frac{a+ib}{c+id} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{ac-adi+bci-bdi^2}{c^2-d^2i} = \frac{(ac+bd)-(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}$

   NOTE: If $z = a + ib$, the *complex conjugate* is

$$\bar{z} = \overline{a + ib} = a - ib$$

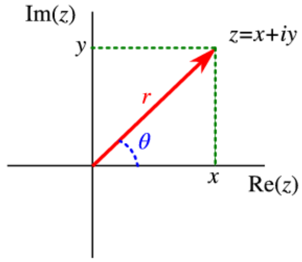
   (b) The Associative and Commutative Properties of addition and multiplication hold.

   (c) The Distributive Property holds.

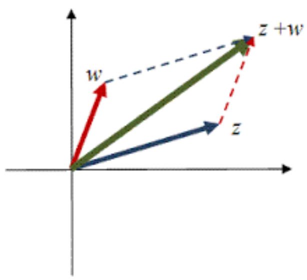
3. Surprisingly the following result is true:
Fundamental Theorem of Algebra: Let $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$ be a polynomial equation with *complex* coefficients. The equation as n complex solutions (counting multiplicity).

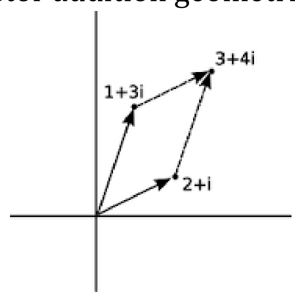
Geometric (Vector) Representation of Complex Numbers:

We can consider $x + iy$ to be an ordered pair (x, y). If we call the x-axis the *real axis*, $\text{Re}(z)$, and the y-axis the *imaginary axis*, $\text{Im}(z)$, we can think of the complex number $z = x + iy$ as a two-dimensional vector.



Addition of complex numbers corresponds to vector addition geometrically.





The length of the vector $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}$$

and is called the *absolute value*, or *modulus*, of the complex number z.

The *distance* d between two complex numbers z_1 and z_2 is

$$d(z_1, z_2) = |z_1 - z_2| = |z_2 - z_1|$$

Properties of Absolute Value:

1. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

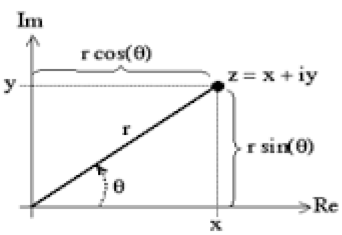
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ when $z_2 \neq 0$

3. $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle Inequality)

4. $|z_1 + z_2| \geq |z_1| - |z_2|$ (Reverse Triangle Inequality)

Polar Form of Complex Numbers:

If $z = x + iy$ and $r = \sqrt{x^2 + y^2} = |z|$, we have

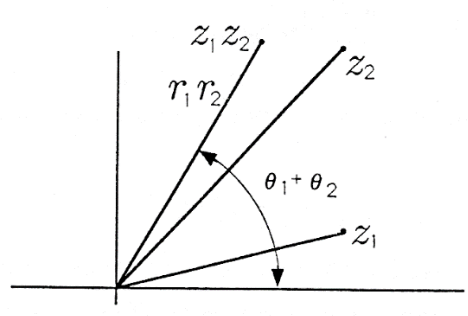


Thus, $z = x + iy = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$. This is called the *polar form* of complex numbers.

Geometric Interpretation of Multiplication:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\begin{aligned} z_1 \cdot z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$



This generalizes to

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n[\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_n)]$$

This result is known as *DeMoirve's Theorem*.

If we define $e^{i\theta} = \cos\theta + i\sin\theta$, we have

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

This is called the *exponential form* of a complex number.

NOTE: At this point $e^{i\theta}$ is just short-hand for $\cos\theta + i\sin\theta$. It can be shown in more advanced classes that this is related to the complex exponential function $f(z) = e^z$.

Using the exponential form of complex numbers, multiplication of complex numbers becomes

$$\begin{aligned} z_1 \cdot z_2 &= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \\ &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$