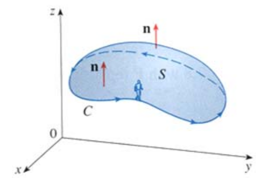
Section 16.8

Stokes' Theorem

DEFINITION

The orientation of a surface S induces the **positive orientation of the boundary curve C** as shown in the diagram. This means that if you walk in the positive direction around C with your head pointing in the direction of $\mathbf{n}$, then the surface will always be on your left.



STOKES' THEOREM

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

NOTE: Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem.

NOTATION

The positively oriented boundary curve of the oriented surface S is often written as ∂S. So the result of Stokes' Theorem can be expressed as

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

COMMENT 1

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

and

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

RELATIONSHIP TO THE FUNDAMENTAL THEOREM OF CALCULUS

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus. There is an integral involving derivatives on the left side of the equation above (recall that curl $\mathbf{F}$ is a sort of derivative) and the right side involves the values of $\mathbf{F}$ only on the *boundary* of S.

GREEN'S THEOREM AS A SPECIAL CASE

The special case where the surface S is flat and lies in the xy-plane with upward orientation, the unit normal is $\mathbf{k}$, the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{k}\,dA$$

This is precisely the vector form of Green's Theorem given in Section 16.5. Thus, we see that Green's Theorem is really a special case of Stokes' Theorem.

EXAMPLES

1. Let ∂S be the triangle formed by the intersection of the plane $2x + 2y + z = 6$ and the three coordinate planes. Verify Stokes's Theorem if $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.
2. Verify Stokes's Theorem for $\mathbf{F}(x, y, z) = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$, where S is the surface of the paraboloid $z = 4 - x^2 - y^2$ and ∂S is the trace of S in the xy-plane.

COMMENT 2

Note that in the second part of Example 2, we computed a surface integral simply by knowing the values of $\mathbf{F}$ on the boundary curve C. This means that if we have any other oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral.

In general if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\iint_{S_1} \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2} \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$$

THE MEANING OF THE CURL VECTOR

Suppose that C is an oriented closed curve and $\mathbf{v}$ represents the velocity field in fluid flow. Consider the line integral

$$\int_C \mathbf{v}\cdot d\mathbf{r} = \int_C \mathbf{v}\cdot\mathbf{T}\,ds$$

and recall that $\mathbf{v}\cdot\mathbf{T}$ is the component of $\mathbf{v}$ in the direction of the unit tangent vector $\mathbf{T}$. This means the closer the direction of $\mathbf{v}$ is to the direction of $\mathbf{T}$, the larger the value of $\mathbf{v}\cdot\mathbf{T}$. Thus, $\int_C \mathbf{v}\cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around C and is called the **circulation** of $\mathbf{v}$ around C.

CURL (CONTINUED)

Let $P_0(x_0, y_0, z_0)$ be a point in the fluid at let S_a be a small disk with radius a and center P_0. Then, $(\text{curl}\,\mathbf{F})(P) \approx (\text{curl}\,\mathbf{F})(P_0)$ for all points P on S_a because the curl of $\mathbf{F}$ is continuous. Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a.

$$\int_{C_a} \mathbf{v}\cdot d\mathbf{r} = \iint_{S_a} \text{curl}\,\mathbf{v}\cdot d\mathbf{S} = \iint_{S_a} \text{curl}\,\mathbf{v}\cdot\mathbf{n}\,dS$$

$$\approx \iint_{S_a} \text{curl}\,\mathbf{v}(P_0)\cdot\mathbf{n}(P_0)dS = \text{curl}\,\mathbf{v}(P_0)\cdot\mathbf{n}(P_0)\pi a^2$$

CURL (CONCLUDED)

The approximation becomes better as $a \to 0$ and we have

$$\text{curl}\,\mathbf{v}(P_0)\cdot\mathbf{n}(P_0) = \lim_{a\to 0}\frac{1}{\pi a^2}\int_{C_a} \mathbf{v}\cdot d\mathbf{r}$$

This gives the relationship between the curl and the circulation. It shows that the curl $\mathbf{v}\cdot\mathbf{n}$ is a measure of the rotating effect of the fluid about the axis $\mathbf{n}$. The curling effect is greatest about the axis parallel to curl $\mathbf{v}$.