# Section 16.4

## Green's Theorem

## POSITIVE ORIENTATION

Let $C$ be a simple closed curve and let $D$ be the plane region enclosed by $C$.

We define the **positive orientation** of a simple closed curve $C$ refers to a single ***counterclockwise*** traversal of $C$.

Thus, if $C$ is given by the vector function $\mathbf{r}(t)$, $a \le t \le b$,then the region $D$ is always on the left as the point $\mathbf{r}(t)$ traverses $C$.

## GREEN'S THEOREM

**Theorem**: Let $C$ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x}\right) dA$$

NOTE: Green's Theorem gives the relationship between a line integral around a simple closed curve $C$ and the double integral over the plane region $D$ bounded by $C$.

## OTHER NOTATIONS FOR THE LINE INTEGRAL

The following notations are also used for the line integral using the ***positive*** orientation of $C$.

$$\oint_C P\,dx + Q\,dy$$

$$\int_{\partial D} P\,dx + Q\,dy$$

Here $\partial$ does not mean partial derivative; it means "boundary of." Thus, $\partial D$ means the boundary of $D$.

## AREA AND LINE INTEGRALS

The area of the simply-connected region $D$ can be computed using any of the following line integrals.

$$A = \oint_C x\,dy$$

$$= -\oint_C y\,dx$$

$$= \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

## AN EXTENSION OF GREEN'S THEOREM

Green's Theorem can be extended to a connected region $D$ with holes. Suppose the region $D$ has only one hole. Observe the boundary $C$ of consists of two simple closed curves $C_1$ and $C_2$. Assume that these curves are oriented so that the region $D$ is always on the left has the curve $C$ is traversed. Divide $D$ into two regions $D'$ and $D''$ by two line segments connecting $C_1$ and $C_2$. We apply Green's Theorem to each of $D'$ and $D''$ and get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

## EXTENSION (CONCLUDED)

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$
$$= \int_C P\,dx + Q\,dy$$

which is Green's Theorem for the region $D$.

This can be extended to connected regions with more than one hole.