STUDY GUIDE FOR IN-CLASS PART OF TEST III
MATH 2201

1. Find the eigenvalues and find bases for the eigenspaces of the following matrices.

(a) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} -7 & -9 & 3 \\ 2 & 4 & -2 \\ -3 & -3 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 0 & 0 \\ \frac{1}{2} & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

2. Determine whether the matrix A is diagonalizable. If it is, find a matrix P that diagonoalizes A, and determine $P^{-1}AP$.

(a) $A = \begin{bmatrix} -9 & -6 & -22 \\ 1 & 2 & 2 \\ 4 & 2 & 10 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 0 & 0 \\ \frac{1}{2} & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

3. Let $\mathbf{u} = (1, 3, -1)$ and $\mathbf{v} = (2, 0, 5)$. Use the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + u_2v_2 + 3u_3v_3$ to compute the following.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle$

(b) $\|\mathbf{u}\|$

(c) $d(\mathbf{u}, \mathbf{v})$

(d) $\cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

4. Let $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$. Use the inner product $\left\langle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$ to compute the following.

(a) $\langle A, B \rangle$

(b) $\|A\|$

(c) $d(A, B)$

(d) $\cos\theta$ where θ is the angle between A and B

5. Let $f(x) = \frac{1}{3} + x^4$ and $g(x) = 6x$. Use the inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)\,g(x)\,dx$ on $C[0,1]$ to compute the following.

(a) $\langle \mathbf{f}, \mathbf{g} \rangle$

(b) $\|\mathbf{f}\|$

(c) $d(\mathbf{f}, \mathbf{g})$

(d) $\cos\theta$ where θ is the angle between $\mathbf{f}$ and $\mathbf{g}$

ANSWERS

1. (a) $\lambda_1 = 0$, basis for eigenspace $B_1 = \{(1, 0, 0)\}$
$\lambda_2 = 1$, basis for eigenspace $B_2 = \{(0, -1, 1)\}$
$\lambda_3 = 2$, basis for eigenspace $B_3 = \{(0, -2, 1)\}$
(b) $\lambda_1 = 2$, basis for eigenspace $B_1 = \{(1, -1, 0)\}$
$\lambda_2 = -2$, basis for eigenspace $B_2 = \{(0, 1, 3)\}$
$\lambda_3 = -4$, basis for eigenspace $B_3 = \{(1, 0, 1)\}$
(c) $\lambda_1 = 2$ (algebraic multiplicity 2), basis for eigenspace $B_1 = \{(-2, 1, 0), (-2, 0, 1)\}$
$\lambda_2 = 3$, basis for eigenspace $B_2 = \{(0, 1, 0)\}$

2. (a) Yes. $P = \begin{bmatrix} -\frac{8}{3} & -\frac{5}{2} & -2 \\ \frac{1}{3} & \frac{1}{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
(b) No, because the algebraic multiplicity of $\lambda = 1$ is 2 but the geometric multiplicity is 1.
(c) Yes. $P = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

3. (a) -11
(b) $\sqrt{14}$
(c) $\sqrt{119}$
(d) $-\dfrac{11}{\sqrt{1162}}$

4. (a) -5
(b) $\sqrt{30}$
(c) 7
(d) $-\dfrac{5}{3\sqrt{30}}$

5. (a) 2
(b) $\dfrac{4}{3\sqrt{5}}$
(c) $\dfrac{376}{45}$
(d) $\dfrac{\sqrt{15}}{4}$