STUDY GUIDE FOR FINAL EXAM
MATH 2201

1. Use Gauss-Jordan elimination to solve (by hand) the following systems of linear equations.

(a) $\begin{aligned} x - y - z &= 0 \\ 2x + y + z &= 3 \\ 3x \quad - z &= 0 \end{aligned}$

(b) $\begin{aligned} 2x_1 + 2x_2 - 2x_3 &= 4 \\ 3x_1 + 5x_2 + x_3 &= -8 \\ -4x_1 - 7x_2 - 2x_3 &= 13 \end{aligned}$

2. Use determinants to decide if the following homogeneous linear systems have a nontrivial solution. Do the computations by hand.

(a) $\begin{aligned} x - 4y + z &= 0 \\ 2x - 3y + 7z &= 0 \\ x - 2y \quad &= 0 \end{aligned}$

(b) $\begin{aligned} x_1 + 7x_2 + x_3 &= 0 \\ 2x_1 + 14x_2 + 5x_3 &= 0 \\ 3x_1 + 21x_2 + 5x_3 &= 0 \end{aligned}$

3. Let **v** = (7, −1, 2, 3) and **w** = (−2, 2, −4, 5). Find the vector **x** that satisfies

$$3(2\mathbf{x} + 7\mathbf{v}) = -2(3\mathbf{w} - 5\mathbf{x}).$$

4. Determine whether the following are matrix operators by using Theorem 4.10.2 which says:
A transformation $T: R^n \to R^m$ is a matrix transformation if and only if the following relationships hold for all vectors **u** and **v** in R^n and for every scalar k.
(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (ii) $T(k\mathbf{u}) = kT(\mathbf{u})$

(a) $T(x, y) = (2x + y, x - y)$

(b) $T(x, y) = (x + 1, y)$

(c) $T(x, y) = (y, y)$

(d) $T(x, y) = \left(\sqrt{x}, \sqrt[3]{y}\right)$

5. The standard matrix $[T]$ of a matrix transform T is given. Use it to find $T(\mathbf{x})$.

(a) $[T] = \begin{bmatrix} 1 & 6 \\ 1 & 7 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $[T] = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 6 & 8 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(c) $[T] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 3 & -2 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(d) $[T] = \begin{bmatrix} 1 & 5 & 9 \\ 6 & 1 & -1 \\ 0 & 2 & 5 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

6. Determine if the following sets W are subspaces of the indicated vector space V.

   (a) $W = \{(a, b, c) \mid c = a - 2b\}$; $V = R^3$

   (b) $W = \{(a, b, c) \mid a^2 + b^2 = c^2\}$; $V = R^3$

   (c) $W = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, \text{ and } a_2 \text{ are integers}\}$; $V = P_2$

7. Determine if the following sets of vectors in R^3 are linearly independent or linearly dependent.

   (a) $\left\{(3, 6, -12), \left(-\frac{2}{5}, -\frac{4}{5}, \frac{8}{5}\right)\right\}$

   (b) $\{(1, -4, 2), (2, 0, 1), (3, 2, 2)\}$

   (c) $\{(1, 1, 1), (1, 1, 0), (1, 0, 0), (1, 0, 1)\}$

8. Determine if the following sets of vectors in R^4 are linearly independent or linearly dependent.

   (a) $\{(1, 2, 4, -3), (1, 1, 0, 1), (2, 1, 1, 3)\}$

   (b) $\{(3, 2, 2, 2), (12, 5, 2, 2), (6, 2, 5, 2), (3, 2, 2, 5)\}$

   (c) $\{(1, a, 0, 0), (0, 1, a, 0), (0, 0, 1, a), (0, 0, 0, 1)\}$

9. Find the dimension of the subspace spanned by the given vectors.

   (a) $\{(1, 3, 2), (1, -5, 5), (3, 1, 9)\}$

   (b) $\{(-1, 2, 2, 3), (2, 1, 1, 5), (7, -4, -4, 1)\}$

   (c) $\{(1, 0, 2, -1), (3, 3, 0, 1), (0, -3, 6, -4), (1, 3, -4, 3)\}$

   (d) $\{(1, 1, -1, 2), (1, -1, 1, 1), (3, -5, 5, 2), (6, 2, -3, 8), (8, 2, -4, 9)\}$

10. Find the distance between the given vecotrs in the given inner product space.

(a) $\mathbf{u} = (1, 3, -1)$, $\mathbf{v} = (2, 0, 4)$. Use the Euclidean inner product on R^3

(b) $\mathbf{u} = (1, -3)$, $\mathbf{v} = (5, 1)$. Use the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 5u_2v_2$.

(c) $\mathbf{p} = 1 + x^2$, $\mathbf{q} = 1 - x^2$. Use the inner product on P_2 given by $\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + a_2b_2$.

11. Use the inner products given in Question 10 to determine if the following pairs of vectors are orthogonal.

(a) $\mathbf{u} = (1, 4, -5)$, $\mathbf{v} = (2, 3, 2)$

(b) $\mathbf{u} = (2, 3)$, $\mathbf{v} = (10, -4)$

(c) $\mathbf{p} = 1 - x^2$, $\mathbf{q} = 3 + 12x - 4x^2$

12. For each of the following matrices, find:
   (i) the characteristic equation of the matrix A
   (ii) the eigenvalues of the matrix A
   (iii) a basis for each eigenspace of the matrix A.

(a) $A = \begin{bmatrix} 4 & -12 \\ -6 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

13. For the matrices given in Question 12, determine if A is diagonalizable. If it is, find the matrix P that diagonalizes A, and determine $P^{-1}AP$.

ANSWERS

1. (a) $x = 1, y = -2, z = 3$ (b) $x_1 = 9 + 3t,\ x_2 = -7 - 2t, x_3 = t$

2. (a) det $= -15 \neq 0$. The system has only the trivial solution.
   (b) det = 0. The system has a nontrivial solution.

3. $\mathbf{x} = \left(\frac{135}{4}, -\frac{9}{4}, \frac{9}{2}, \frac{93}{4}\right)$

4. (a) matrix transform (b) not a matrix transform
   (c) matrix transform (d) not a matrix transform

5. (a) $\begin{bmatrix} 13 \\ 15 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$
   (c) $\begin{bmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \\ 3x_1 - 2x_2 \end{bmatrix}$ (d) $\begin{bmatrix} x_1 + 5x_2 + 9x_3 \\ 6x_1 + x_2 - x_3 \\ 2x_2 + 5x_3 \end{bmatrix}$

6. (a) subspace (b) not a subspace
   (c) not a subspace

7. (a) linearly dependent (b) linearly independent
   (c) linearly dependent

8. (a) linearly independent (b) linearly independent
   (c) linearly independent

9. (a) two dimensional (b) two dimensional
   (c) two dimensional (d) three dimensional

10. (a) $\sqrt{35}$ (b) $8\sqrt{2}$
    (c) 2

11. (a) not orthogonal (b) orthogonal
    (c) not orthogonal

12. (a) $\lambda^2 - 7\lambda - 60$; $\lambda = 12, -5$. For $\lambda = 12$, $\{(-3, 2)\}$. For $\lambda = -5$, $\{(4, 3)\}$.

(b) $(\lambda - 2)(\lambda - 1)^2$; $\lambda = 1, 1, 2$. For $\lambda = 1$, $\{($[illegible]$, \frac{4}{3}, 1)\}$. For $\lambda = 2$, $\{($[illegible]$, \frac{3}{4}, 1)\}$.

(c) $(\lambda - 1)^2(\lambda - 2)^2$; $\lambda = 1, 1, 2, 2$. For $\lambda = 1$, $\{(1, -1, 0, 0), (1, 0, 1, 0)\}$. For $\lambda = 2$ $\{(1, 0, 2, 0), (-1, 0, 0, 2)\}$.

13. (a) $P = \begin{bmatrix} -3 & 4 \\ 2 & 3 \end{bmatrix}$; $P^{-1}AP = \begin{bmatrix} 12 & 0 \\ 0 & -5 \end{bmatrix}$

(b) Not diagonalizable.

(c) $P = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$; $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$