Section 7.1

Orthogonal Matrices

ORTHOGONAL MATRICES

A square matrix A is said to be **orthogonal** if its transpose is the same as its inverse; that is,

$$A^{-1} = A^T$$

or, equivalently, if

$$A^TA = AA^T = I.$$

THEOREM

Theorem 7.1.1: The following are equivalent for an $n \times n$ matrix A.

(a) A is orthogonal.

(b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.

(c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

PROPERTIES OF ORTHOGONAL MATRICES

Theorem 7.1.2:

(a) The inverse of an orthogonal matrix is orthogonal.

(b) A product of orthogonal matrices is orthogonal.

(c) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

ORTHOGONAL MATRICES AS LINEAR OPERATORS

Theorem 7.1.3: If A is an $n \times n$ matrix, the following are equivalent.

(a) A is orthogonal

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in R^n.

THEOREM

Theorem 7.1.4: If S is an orthonormal basis for an n-dimensional inner product space V, and if

$$(\mathbf{u})_S = (u_1, u_2, \dots, u_n) \text{ and } (\mathbf{v})_S = (v_1, v_2, \dots, v_n)$$

then:

(a) $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

(b) $d(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$

(c) $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

ORTHOGONAL LINEAR OPERATORS

If $T: R^n \rightarrow R^n$ is multiplication by an orthogonal matrix A, then T is called an **orthogonal linear operator**.

Note: It follows from parts (a) and (b) of the preceding theorem that orthogonal linear operators are those operators that leave the length of vectors unchanged.

CHANGE OF ORTHONORMAL BASIS

Theorem 7.1.5: Let V be a finite-dimensional inner product space. If P is the transition matrix from one orthonormal basis to another orthonormal basis for an inner product space, then P is an orthogonal matrix; that is,

$$P^{-1} = P^T$$