Section 6.3

Orthonormal Bases;
Gram-Schmidt Process;
***QR*-Decomposition**

ORTHONORMAL SETS OF VECTORS

A set of two or more vectors in a real inner product space is said to be **orthogonal** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is called **orthonormal**.

ORTHOGONALITY AND INDEPENDENCE

Theorem 6.3.1: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

ORTHOGONAL AND ORTHONORMAL BASES

- In an inner product space, a basis consisting of orthogonal vectors is called an **orthogonal basis**.
- In an inner product space, a basis consisting of orthonormal vectors is called an **orthonormal basis**.

COORDINATES RELATIVE TO AN ORTHONORMAL BASES

Theorem 6.3.2:

(a) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n\rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

(b) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_n\rangle \mathbf{v}_n$$

COORDINATE VECTORS

Let $\mathbf{u}$ be any vector in an inner product space V. The coordinate vector of $\mathbf{u}$ relative to an orthogonal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

$$(\mathbf{u})_S = \left(\frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}, \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}, \cdots, \frac{\langle \mathbf{u}, \mathbf{v}_n\rangle}{\|\mathbf{v}_n\|^2}\right)$$

and relative to an orthonomal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is

$$(\mathbf{u})_S = \left(\langle \mathbf{u}, \mathbf{v}_1\rangle, \langle \mathbf{u}, \mathbf{v}_2\rangle, \cdots, \langle \mathbf{u}, \mathbf{v}_n\rangle\right)$$

THE PROJECTION THEOREM

Theorem 6.3.3: If W is a finite-dimensional subspace of an inner product space V, then every vector $\mathbf{u}$ in V can be expressed inexactly one way as

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ in $W^{\perp}$.

NOTE: $\mathbf{w}_1$ is called the **orthogonal projection of u on *W*** and is denoted by $\text{proj}_W\, \mathbf{u}$. The vector $\mathbf{w}_2$ is called the **complement of u orthogonal to *W*** and is denoted by $\text{proj}_{W^{\perp}}\, \mathbf{u}$.

THEOREM

Theorem 6.3.4: Let W be a finite-dimensional subspace of an inner product space V.

(a) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W\, \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r$$

(b) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W\, \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r$$

THEOREM

Theorem 6.3.5: Every nonzero finite-dimensional inner product space has an orthonormal basis.

THE GRAM-SCHMIDT PROCESS

To convert a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$ into an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, perform the following computions.

Step 1: $\mathbf{v}_1 = \mathbf{u}_1$.

Step 2: $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

Step 3: $\mathbf{v}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

G-S PROCESS (CONCLUDED)

Step 4: $\mathbf{v}_4 = \mathbf{u}_4 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \dfrac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$

Continue the process for r steps until all $\mathbf{u}_i$ are exhausted.

Optional Step: To convert the orthogonal basis into an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_r\}$, normalize the orthogonal basis vectors.