Section 6.2

Angles and Orthogonality in Inner Product Spaces

CAUCHY-SCHWARZ INEQUALITY

Theorem 6.2.1: If **u** and **v** are vectors in a real inner product space, then

$$\left|\langle \mathbf{u}, \mathbf{v} \rangle\right| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

NOTE: This result can be written as

$$\langle \mathbf{u}, \mathbf{v} \rangle^2 \le \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle$$

$$\langle \mathbf{u}, \mathbf{v} \rangle^2 \le \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2$$

ANGLE BETWEEN VECTORS

Recall in R^2 and R^3, we noted that if θ is the angle between two vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

For an arbitrary inner product space we ***define*** the **angle θ between u and v** between two vectors **u** and **v** to be

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$$

PROPERTIES OF LENGTH AND DISTANCE

Theorem 6.2.2: If **u** and **v** are vectors in an inner product space V, and if k is a scalar, then:

(a) $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

[Triangle inequality for vectors]

(b) $d(\mathbf{u}, \mathbf{v}) \le d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$

[Triangle inequality for distance]

ORTHOGONALITY

Two vectors **u** and **v** in an inner product space are called **orthogonal** if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

GENERALIZED THEOREM OF PYTHAGORAS

Theorem 6.2.3: If **u** and **v** are orthogonal vectors in an inner product space, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

ORTHOGONAL COMPLEMENTS

Let W be a subspace of an inner product space V. A vector **u** in V is said to be **orthogonal to *W*** if it is orthogonal to every vector in W. The set of all vectors in V that are orthogonal to W is called the **orthogonal complement** of W.

Notation:
We denote the orthogonal complement of a subspace W by $W^\perp$. [read "W perp"]

PROPERTIES OF ORTHOGONAL COMPLEMENTS

Theorem 6.2.4: If W is a subspace of an inner product space V, then

(a) $W^\perp$ is a subspace of V.

(b) $W \cap W^\perp = [\mathbf{0}]$.

ANOTHER PROPERTY OF ORTHOGONAL COMPLEMENTS

Theorem 6.2.5: If W is a subspace of a finite-dimensional inner product space V, then the orthogonal complement of $W^\perp$ is W; that is,

$$(W^\perp)^\perp = W.$$