Section 6.1

Inner Products

INNER PRODUCT SPACES

An **inner product** on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors **u** and **v** in V in such a way that the following axioms are satisfied for all vectors **u**, **v**, and **w** in V and all scalars k.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [**Symmetry Axiom**]

(b) $\langle \mathbf{u} + \mathbf{v}, \mathbf{z} \rangle = \langle \mathbf{u}, \mathbf{z} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle$ [**Additivity Axiom**]

(c) $\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ [**Homogeneity Axiom**]

(d) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ [**Positivity Axiom**]

A real vector space with an inner product is called a **real inner product space**.

NORM AND DISTANCE

If V is an inner product space, then the **norm** (or **length**) of a vector **u** in V is denoted by $\|\mathbf{u}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

The **distance** between two vectors (points) **u** and **v** is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

WEIGHTED EUCLIDEAN INNER PRODUCT

If $w_1, w_2, \ldots, w_n$ are positive real numbers, called **weights**, and if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then it can be shown that the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

defines an inner product on R^n; it is called the **weighted Euclidean inner product with weights $w_1, w_2, \ldots, w_n$**.

PROPERTIES OF NORM AND DISTANCE

Theorem 6.1.1: If **u** and **v** are vectors in a real inner product space V, and if k is

(a) $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{0}$.

(b) $\|k\mathbf{v}\| = |k| \, \|\mathbf{v}\|$.

(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.

(d) $d(\mathbf{u}, \mathbf{v}) \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{0}$.

THE UNIT SPHERE

The set of all points (vectors) in V that satisfy $\|\mathbf{u}\| = 1$ is called the **unit sphere** (or sometimes the **unit circle**) in V. In R^2 and R^3 these are points that lie 1 unit, in terms of the inner product, away from the origin. If you are using a different inner product than the dot product (*e.g.*, the weighted Euclidean inner product), the "unit circle" may not be a circle!

PROPERTIES OF INNER PRODUCTS

Theorem 6.1.2: If **u**, **v**, and **w** are vectors in a real inner product space, and *k* is any scalar, then

(a) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

(b) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(c) $\langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$

(d) $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$

(e) $k\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, k\mathbf{v} \rangle$