Section 5.2

Diagonalization

TWO PROBLEMS

- **The Diagonalization Problem:** Given an $n \times n$ matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix?
- **The Eigenvector Problem:** Given an n n matrix A, does A have n linear independent eigenvectors?

SIMILAR MATRICES

If A and B are square matrices, then we say that ***B* is similar to *A*** if there is an invertible matrix P such that $B = P^{-1}AP$.

DIAGONALIZABLE MATRICES

A square matrix A is called **diagonalizable** if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix; the matrix P is said to **diagonalize** A.

TWO EQUIVALENT STATEMENTS

Theorem 5.2.1: If A is an $n \times n$ matrix, then the following are equivalent.

(a) A is diagonalizable.

(b) A has n linearly independent eigenvectors.

PROCEDURE TO DIAGONALIZE A MATRIX

Step 1: Confirm that the matrix is actually diagonalizable by finding n linearly independent eigenvectors. One way to do this is by finding a basis for each eigenspace and merging these basis vectors into a single set S. If this set has fewer than n vectors , then the matrix is not diagonalizable.

Step 2: Form the matrix $P = [\mathbf{p}_1 \ \ \mathbf{p}_2 \ \ \ldots \ \ \mathbf{p}_n]$ that has the vectors in S as it column vectors.

Step 3: The matrix $P^{-1}AP$ will then be diagonal and have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ corresponding to the eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots \ \mathbf{p}_n$ as its successive entries.

THEOREM

Theorem 5.2.2: If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors for A corresponding to ***distinct*** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set.

THEOREM

Theorem 5.2.3: If an $n \times n$ matrix A has n distinct eigenvectors, then A is diagonalizable.

GEOMETRIC AND ALGEBRAIC MULTIPLICITY

- If λ_0 is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the **geometric multiplicity** of λ_0.
- The number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the **algebraic multiplicity** of A.

THEOREM 5.2.5 GEOMETRIC AND ALGEBRAIC MULTIPLICITY

Theorem 5.2.5: If A is a square matrix, then:

(a) For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.

(b) A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.