Section 5.1

Eigenvalues and Eigenvectors

EIGENVALUES AND EIGENVECTORS

If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an **eigenvector** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A (or of T_A), and $\mathbf{x}$ is said to be the **eigenvector corresponding to λ**.

CHARACTERISTIC EQUATION AND CHARACTERISTIC POLYNOMIAL

Theorem 5.1.1: If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0.$$

This is called the **characteristic equation**. After the determinant is expanded, it is a polynomial in λ which is called the **characteristic polynomial**.

EIGENVALUES OF TRIANGULAR MATRICES

Theorem 5.1.2: If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

EQUIVALENT STATEMENTS

Theorem 5.1.3: If A is an $n \times n$ matrix and λ is a real number, then the following are equivalent.

(a) λ is an eigenvalue of A.

(b) The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

(c) There is an nonzero vector $\mathbf{x}$ in R^n such that $A\mathbf{x} = \lambda\mathbf{x}$.

(d) λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.

EIGENSPACES

The eigenvectors corresponding the λ are the nonzero vectors in the solution space of $(\lambda I - A)\mathbf{x} = \mathbf{0}$. We call this null space the **eigenspace** of A corresponding to λ.

POWERS OF A MATRIX

Theorem 5.1.4: If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

EIGENVALUES AND INVERTIBILITY

Theorem 5.1.5: A square matrix A is invertible if and only $\lambda = 0$ is ***not*** an eigenvalue of A.

THE "BIG" THEOREM

Theorem 4.10.4: If A is an $n\times n$ matrix, then the following are equivalent.

(a) A is invertible

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row-echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n\times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n\times 1$ matrix $\mathbf{b}$.

(g) $\det(A) \neq 0$

(h) The column vectors of A are linearly independent.

(i) The row vectors of A are linearly independent.

THE "BIG" THEOREM (CONCLUDED)

(j) The column vectors of A span R^n.

(k) The row vectors of A span R^n.

(l) The column vectors of A form a basis for R^n.

(m) The row vectors of A form a basis for R^n.

(n) A has rank n.

(o) A has nullity 0.

(p) The orthogonal complement of the null space of A is R^n.

(q) The orthogonal complement of the row space of A is $[\mathbf{0}]$.

(r) The range of T_A is R^n.

(s) T_A is one-to-one.

(t) $\Lambda = 0$ is not an eigenvalue of A.