Section 4.9

Matrix Transformations from R^n to R^m

TRANSFORMATIONS

If V and W are vector spaces, and if f is a function with domain V and codomain W, then we say that f is a **transformation** from V to W or that f **maps** V to W, which we denote by writing

$$f: V \to W$$

In the special case where $V = W$, the transformation is also called an **operator** on V.

TRANSFORMATIONS FROM R^n TO R^m

Let $f_1, f_2, \dots, f_m$ be real-valued functions of n variables, say

$$\begin{aligned} w_1 &= f_1(x_1, x_2, \dots, x_n) \\ w_2 &= f_2(x_1, x_2, \dots, x_n) \\ &\vdots \\ w_m &= f_m(x_1, x_2, \dots, x_n) \end{aligned}$$

These equations assign a unique point $(w_1, w_2, \dots w_n)$ in R^m and define a transformation from R^n to R^m.

NOTATION AND MATRIX TRANSFORMATIONS

If we denote the transformation by T, then

$$T : R^n \to R^m \quad \text{and}$$

$$T(x_1, x_2, \dots, x_n) = (w_1, w_2, \dots, w_m)$$

If the equations are linear, the transformation $T: R^n \to R^m$ is called a **linear transformation** (or **linear operator** if $m = n$).

STANDARD MATRIX FOR A MATRIX TRANSFORMATION

Let $T: R^n \to R^m$ and $T(x_1, x_2, \dots, x_n) = (w_1, w_2, \dots, w_n)$ where $w_i = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$ for $1 \le i \le m$.

In matrix notation,

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or $\mathbf{w} = A\mathbf{x}$.

We call this a **matrix transform** (or **matrix operator** if $m = n$), and we denote it by $T_A: R^n \to R^m$.

SOME NOTATION AND TERMINOLOGY

- We can express the matrix transform as

$$T_A(\mathbf{x}) = A\mathbf{x}$$

- The matrix transform T_A is called **multiplication by A**, and the matrix A is called the **standard matrix** for the transformation.

BASIC PROPERTIES OF MATRIX TRANSFORMS

Theorem 4.9.1: For every matrix A the matrix transform $T_A: R^n \to R^m$ has the following properties for all vectors **u** and **v** in R^n and for every scalar k:

(a) $T_A(\mathbf{0}) = \mathbf{0}$

(b) $T_A(k\mathbf{u}) = k\,T_A(\mathbf{u})$ **[Homogeneity property]**

(c) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ **[Additivity property]**

(d) $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

EQUALITY OF TRANSFORMS

Theorem 4.9.2: If $T_A: R^n \to R^m$ and $T_B: R^n \to R^m$ are matrix transforms, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector **x** in R^n, then $A = B$.

FINDING THE STANDARD MATRIX FOR A MATRIX TRANSFORM

Step 1: Find the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for R^n in column form.

Step 2: Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transform.

GEOMETRY OF MATRIX TRANSFORMATIONS

The geometry of linear transformation is given in the Tables 4.2.1 through 4.2.10 on pages 252-258.