Section 4.8

Rank and Nullity

ROW SPACE AND COLUMN SPACE HAVE EQUAL DIMENSION

Theorem 4.8.1: The row space and column space of a matrix A have the same dimension.

RANK AND NULLITY

The common dimension of the row space and column space of a matrix A is called the **rank** of A and is denoted by rank(A). The dimension of the null space of A is called the **nullity** of A and is denoted by nullity(A).

DIMENSION THEOREM FOR MATRICES

Theorem 4.8.2: If A is any matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

THEOREM

Theorem 4.8.3: If A is an $m \times n$ matrix, then:

(a) rank(A) = the number of leading variables in the solution of $A\mathbf{x} = \mathbf{0}$.

(b) nullity(A) = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

MINIMUM VALUE FOR RANK

If A is an $m \times n$ matrix, then the row vectors lie in R^n and the column vectors in R^m. This means the row space is at most n-dimensional and the column space is at most m-dimensional. Thus,

$$\text{rank}(A) \leq \min(m, n).$$

THE "BIG" THEOREM

Theorem 4.8.4: If A is an $n \times n$ matrix, then the following are equivalent.

(a) A is invertible

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row-echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(g) $\det(A) \neq 0$

(h) The column vectors of A are linearly independent.

THE "BIG" THEOREM (CONCLUDED)

(i) The row vectors of A are linearly independent.

(j) The column vectors of A span R^n.

(k) The row vectors of A span R^n.

(l) The column vectors of A form a basis for R^n.

(m) The row vectors of A form a basis for R^n.

(n) A has rank n.

(o) A has nullity 0.

OVERDETERMINED AND UNDERDETERMINED SYSTEMS

- Systems with more constraints than unknowns are called **overdetermined systems**.
- Systems with fewer constraints than unknowns are called **underdetermined systems**.

PARAMETERS AND RANK

Theorem 4.8.5: If $A\mathbf{x} = \mathbf{b}$ is a consistent linear systems of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

A THEOREM

Theorem 4.8.5: Let A be an $n \times n$ matrix.

(a) (***Overdetermined Case***). If $m > n$, then the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent for at least one vector $\mathbf{b}$ in R^n.

(b) (***Underdetermined Case***). If $m < n$, then for each vector $\mathbf{b}$ in R^n the linear system $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.

FOUR FUNDAMENTAL MATRIX SPACES

If we consider matrices A and A^T together, then there are six vector spaces of interest:

row space A	row space A^T
column space A	column space A^T
nullspace A	nullspace A^T.

Since transposing converts row vectors to column vectors and vice versa, we really only have four vector spaces of interest:

row space A	column space A
nullspace A	nullspace A^T

These are known as the **fundamental matrix spaces** associated with A.

RANK OF A MATRIX AND ITS TRANSPOSE

Theorem 4.8.7: If A is any matrix, then

$$\text{rank}(A) = \text{rank}(A^T).$$

SOME RELATIONSHIPS

Let A be an $m \times n$ matrix.

$$\text{rank}(A) + \text{nullity}(A^T) = m$$

$\dim[\text{row}(A)] = r$ $\quad$ $\dim[\text{col}(A)] = r$

$\dim[\text{null}(A)] = n - r$ $\quad$ $\dim[\text{null}(A^T)] = m - r$

ORTHOGONAL COMPLEMENTS

If W is a subspace of R^n, then the set of all vectors in R^n that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^\perp$.

A THEOREM

Theorem 4.8.8: If W is a subspace of R^n, then:

(a) $W^\perp$ is a subspace of R^n.

(b) The only vector common to W and $W^\perp$ is $\mathbf{0}$.

(c) The orthogonal complement of $W^\perp$ is W.

A THEOREM

Theorem 4.8.9: If A is an $m \times n$ matrix, then:

(a) The null space of A and the row space of A are orthogonal complements in R^n.

(b) The null space of A^T and the column space of A are orthogonal complements in R^n.

THE "BIG" THEOREM

Theorem 4.8.10: If A is an $n \times n$ matrix, then the following are equivalent.

(a) A is invertible

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row-echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(g) $\det(A) \neq 0$

(h) The column vectors of A are linearly independent.

THE "BIG" THEOREM (CONCLUDED)

(i) The row vectors of A are linearly independent.

(j) The column vectors of A span R^n.

(k) The row vectors of A span R^n.

(l) The column vectors of A form a basis for R^n.

(m) The row vectors of A form a basis for R^n.

(n) A has rank n.

(o) A has nullity 0.

(p) The orthogonal complement of the null space of A is R^n.

(q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.