Section 4.7

Row Space, Column Space, and Null Space

ROW VECTORS

For an $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

the vectors

$$\mathbf{r}_1 = [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}]$$
$$\mathbf{r}_2 = [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}]$$
$$\vdots$$
$$\mathbf{r}_m = [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}]$$

in R^n formed from the rows of A are called the **row vectors** of A.

COLUMN VECTORS

For the $m \times n$ matrix on the previous slide, the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m formed from the columns of A are called the **column vectors** of A.

ROW SPACE, COLUMN SPACE, NULL SPACE

If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the **row space** of A, and the subspace of R^m spanned by the column vectors is called the **column space** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is called the **null space** of A.

TWO QUESTIONS

- What relationships exist among the solutions of a linear system $A\mathbf{x} = \mathbf{b}$ and the row space, column space, and null space of the coefficient matrix A?
- What relationships exists among the row space, column space, and null space of a matrix?

THEOREM

Theorem 4.7.1: A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

THEOREM

Theorem 4.7.2: If $\mathbf{x}_0$ denotes any single solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ form a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

PARTICULAR AND GENERAL SOLUTIONS

- The vector $\mathbf{x}_0$ is called a **particular solution** of $A\mathbf{x} = \mathbf{b}$.
- The expression

  $$\mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

  is called the **general solution** of $A\mathbf{x} = \mathbf{b}$.
- The expression

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

  is called the **general solution** of $A\mathbf{x} = \mathbf{0}$.

THEOREMS ON ROW SPACE AND NULLSPACE

- **Theorem 4.7.3:** Elementary row operations do not change the nulls pace of a matrix.
- **Theorem 4.7.4:** Elementary row operations do not change the row space of a matrix.

A ROW SPACE AND COLUMN SPACE THEOREM

Theorem 4.7.5: If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

A THEOREM ON ROW EQUIVALENT MATRICES

Theorem 4.7.6: If A and B are row equivalent matrices, then:

(a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors in B are linearly independent.

(b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.