### Section 4.6

Change of Basis

#### **COORDINATE MAPS**

If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a basis for a finite-dimensional vector space V, and if

$$(\mathbf{v})_{S} = (c_1, c_2, \ldots, c_n)$$

is the coordinate of vector  $\mathbf{v}$  relative to S, then the mapping

$$\mathbf{v} \rightarrow (\mathbf{v})_{s}$$

creates a one-to-one correspondence between the vectors in the *general* vector space V and vectors in the *familiar* vector space  $R^n$ . We call this mapping the coordinate map from V to  $R^n$ .

#### **CHANGE OF BASIS PROBLEM**

If  $\mathbf{v}$  is a vector in a finite-dimensional vector space V, and if we change the basis for V from a basis B to a basis B', how are the coordinate vectors  $[\mathbf{v}]_B$  and  $[\mathbf{v}]_{B'}$  related?

## SOLUTION TO THE CHANGE OF BASIS PROBLEM

If we change the basis for a vector space V from an old basis  $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  to a new basis  $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n\}$ , then for each vector  $\mathbf{v}$  in V, the old coordinate matrix  $[\mathbf{v}]_B$  of a vector  $\mathbf{v}$  is related to the new coordinate matrix  $[\mathbf{v}]_B$  of  $\mathbf{v}$  by the equation

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$$

where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is, the column vectors of P are

$$[\mathbf{u}_{1}']_{R}, \ [\mathbf{u}_{2}']_{R}, ..., [\mathbf{u}_{n}']_{R}$$

## TRANSITION MATRIX

The matrix *P* is called the <u>transition matrix</u> from *B'* to *B*; it can be expressed in terms of its column vectors as

$$P_{B'\to B} = [\mathbf{u}'_1]_B | [\mathbf{u}'_2]_B | \cdots | [\mathbf{u}'_n]_B^{-1}$$

Similarly, the transition matrix from *B* to *B'* can be expressed in terms of its column vectors as

$$P_{B\to B'} = [\mathbf{u}_1]_{B'} | [\mathbf{u}_2]_{B'} | \cdots | [\mathbf{u}_n]_{B'}^{-1}$$

#### REMARK

$$\begin{bmatrix} \mathbf{v} \end{bmatrix}_B = P_{B' \to B} \begin{bmatrix} \mathbf{v} \end{bmatrix}_{B'}$$
  
 $\begin{bmatrix} \mathbf{v} \end{bmatrix}_{B'} = P_{B \to B'} \begin{bmatrix} \mathbf{v} \end{bmatrix}_{B}$ 

#### **THEOREM**

**Theorem 4.6.1:** If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible and  $P^{-1}$  is the transition matrix from B to B'.

#### **SUMMARY**

$$\begin{bmatrix} \mathbf{v} \end{bmatrix}_B = P[\mathbf{v}]_{B'}$$
  
 $\begin{bmatrix} \mathbf{v} \end{bmatrix}_{B'} = P^{-1}[\mathbf{v}]_{B}$ 

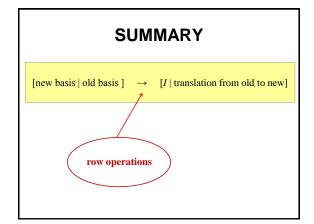
# PROCEDURE FOR COMPUTING $P_{B \rightarrow B'}$

**Step 1:** Form the matrix [B' | B].

Step 2: Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

**Step 3:** The resulting matrix will be  $[I \mid P_{B \to B'}]$ .

**Step 4:** Extract the matrix  $P_{B \to B'}$  from the right side of the matrix in Step 3.



#### **THEOREM**

**Theorem 4.6.2:** Let  $B' = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  be any basis for the vector space  $R^n$  and let  $S = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be the standard basis for  $R^n$ . If the vectors in these bases are written in column form then

$$P_{B' \rightarrow S} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \dots \mid \mathbf{u}_n]$$