Section 4.6

Change of Basis

COORDINATE MAPS

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a finite-dimensional vector space V, and if

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

is the coordinate of vector **v** relative to S, then the mapping

$$\mathbf{v} \rightarrow (\mathbf{v})_S$$

creates a one-to-one correspondence between the vectors in the *general* vector space V and vectors in the *familiar* vector space R^n. We call this mapping the **coordinate map** from V to R^n.

CHANGE OF BASIS PROBLEM

If **v** is a vector in a finite-dimensional vector space V, and if we change the basis for V from a basis B to a basis B', how are the coordinate vectors $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?

SOLUTION TO THE CHANGE OF BASIS PROBLEM

If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$, then for each vector **v** in V, the old coordinate matrix $[\mathbf{v}]_B$ of a vector **v** is related to the new coordinate matrix $[\mathbf{v}]_{B'}$ of **v** by the equation

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$$

where the columns of P are the coordinate vectors of the new basis vectors relative to the old basis; that is, the column vectors of P are

$$[\mathbf{u}'_1]_B,\ [\mathbf{u}'_2]_B, \ldots, [\mathbf{u}'_n]_B$$

TRANSITION MATRIX

The matrix P is called the **transition matrix** from B' to B; it can be expressed in terms of its column vectors as

$$P_{B' \rightarrow B} = \left[[\mathbf{u}'_1]_B \mid [\mathbf{u}'_2]_B \mid \cdots \mid [\mathbf{u}'_n]_B \right]$$

Similarly, the transition matrix from B to B' can be expressed in terms of its column vectors as

$$P_{B \rightarrow B'} = \left[[\mathbf{u}_1]_{B'} \mid [\mathbf{u}_2]_{B'} \mid \cdots \mid [\mathbf{u}_n]_{B'} \right]$$

REMARK

$$[\mathbf{v}]_B = P_{B' \rightarrow B}[\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'} = P_{B \rightarrow B'}[\mathbf{v}]_B$$

THEOREM

Theorem 4.6.1: If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B to B'.

SUMMARY

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$$

PROCEDURE FOR COMPUTING $P_{B \to B'}$

Step 1: Form the matrix $[B' \mid B]$.

Step 2: Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3: The resulting matrix will be $[I \mid P_{B \to B'}]$.

Step 4: Extract the matrix $P_{B \to B'}$ from the right side of the matrix in Step 3.

SUMMARY

[new basis | old basis] $\rightarrow$ [I | translation from old to new]

row operations

THEOREM

Theorem 4.6.2: Let $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be any basis for the vector space R^n and let $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for R^n. If the vectors in these bases are written in column form then

$$P_{B' \to S} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \ldots \mid \mathbf{u}_n]$$