Section 4.5

Dimension

FINITE DIMENSIONAL; INFINITE DIMENSIONAL

A nonzero vector space V is called **finite-dimensional** if it contains a finite set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ that forms a basis. If no such set exists, V is called **infinite-dimensional**. In addition, we shall regard the zero vector space to be finite-dimensional.

TWO THEOREMS CONCERNING DIMENSION

Theorem 4.5.1: All bases for a finite-dimensional vector space have the same number of vectors.

Theorem 4.5.2: Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis.

(a) If a set has more than n vectors, then it is linearly dependent.

(b) If a set has fewer than n vectors, then it does not span V.

DIMENSION OF A FINITE-DIMENSIONAL VECTOR SPACE

The **dimension** of a finite-dimensional vector space V is denotes by dim(V) and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

PLUS/MINUS THEOREM

Theorem 4.5.3: Let S be a nonempty set of vectors in a vector space V.

(a) If S is linearly independent, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.

(b) If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,

$$\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$$

THEOREM

Theorem 4.5.4: Let V be an n-dimensional vector space, and let S be a set in V with ***exactly*** n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.

THEOREM

Theorem 4.5.5: Let S be a finite set of vectors in a finite-dimensional vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

THEOREM

Theorem 4.5.6: If W is a subspace of a finite-dimensional vector space V, then:

(a) W is finite dimensional.

(b) $\dim(W) \leq \dim(V)$.

(c) $W = V$ if and only if $\dim(W) = \dim(V)$.