Section 4.3

Linear Independence

LINEAR INDEPENDENCE AND LINEAR DEPENDENCE

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a nonempty set of vectors in a vector space V, then the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$$

has at least one solution, namely,

$$k_1 = 0, \quad k_2 = 0, \quad \ldots, \quad k_r = 0$$

We call this the **trivial solution**. If this is the only solution, then S is said to be a **linearly independent set**. If there are solutions in addition to the trivial solution, then S is said to be a **linearly dependent set**.

AN INDEPENDENCE / DEPENDENCE THEOREM

Theorem 4.3.1: A set S with two or more vectors is:

(a) Linearly dependent if and only if at least one of the vectors of S is expressible as a linear combination of the other vectors in S.

(b) Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.

THEOREM

Theorem 4.3.2:

(a) A finite set of vectors that contains $\mathbf{0}$ is linearly dependent.

(b) A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.

(c) A set with exactly two vectors in linearly independent if and only if neither vector is a scalar multiple of the other.

GEOMETRIC INTERPRETATION OF LINEAR INDEPENDENCE

- In R^2 or R^3, two vectors are linearly independent if and only if the vectors do not lie on the same line when they are placed with their initial points at the origin.
- In R^3, a set of three vectors is linearly independent if and only if the vectors do not lie in the same plane when they are placed with their initial points at the origin.

LINEAR INDEPENDENCE AND R^n

Theorem 4.3.3: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

THE WRONSKIAN

If $\mathbf{f}_1 = f_1(x)$, $\mathbf{f}_2 = f_2(x)$, . . . , $\mathbf{f}_n = f_n(x)$ are n − 1 times differentiable functions on the interval (−∞, ∞), then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of $f_1, f_2, \ldots, f_n$.

THEOREM

Theorem 4.3.4: If $\mathbf{f}_1 = f_1(x)$, $\mathbf{f}_2 = f_2(x)$, . . . , $\mathbf{f}_n = f_n(x)$ are n − 1 times differentiable functions on the interval (−∞, ∞) and if the Wronksian of these functions is not identically zero on (−∞, ∞), then these functions form a linearly independent set of vectors in $C^{(n-1)}(-\infty, \infty)$.