Section 4.2

Subspaces

SUBSPACES

A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

DETERMINING IF *W* IS A SUBSPACE

Theorem 4.2.1: If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W.

(b) If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.

Remarks:

- If condition (a) holds, W is said to be **closed under addition**.
- If condition (b) holds, W is said to be **closed under scalar multiplication**.
- Thus, W is a subspace if and only if W is closed under addition and closed under scalar multiplication.

INTERSECTION OF SUBSPACES

Theorem 4.2.2: If $W_1, W_2, \ldots, W_r$ are subspaces of a vector spaces V, then the intersection of these subspaces is also a subspace of V.

LINEAR COMBINATIONS

A vector $\mathbf{w}$ is a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

SMALLEST SUBSPACE CONTAINING VECTORS

Theorem 4.2.3: If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ are vectors in a vector space V, then:

(a) The set W of all linear combinations of the vectors in S is a subspace of V.

(b) The set W in part (a) is the smallest subspace of V that contains all of the vectors in S in the sense that every other subspace of V that contains those vectors must contain W.

SPAN

The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the **span of S**, and we say that the vectors in S **span** that subspace. If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$, then we denote the span of S by

$$\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \quad \text{or} \quad \text{span}(S)$$

SOLUTION SPACES OF HOMOGENEOUS SYSTEMS

Theorem 4.2.4: The solution of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in n unknowns is a subspace of R^n.

A THEOREM

Theorem 4.2.5: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then

$$\text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \text{span}\,\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.