Section 4.10

Properties of Matrix Transforms

COMPOSITION OF MATRIX TRANSFORMATIONS

If T_A: $R^n \to R^k$ and T_B: $R^k \to R^m$ are linear transformations, then the application of T_A followed by T_B produces a transformation from R^n to R^m. This transformation is called the **composition of T_B with T_A**, and is denoted by $T_B \circ T_A$. Thus,

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$$

$T_B \circ T_A$ IS A MATRIX TRANSFORM

The composition $T_B \circ T_A$ is a matrix transform since

$$\begin{aligned}(T_B \circ T_A)(\mathbf{x}) &= T_B(T_A(\mathbf{x})) \\ &= B(A\mathbf{x}) \\ &= (BA)\mathbf{x}\end{aligned}$$

The above formula also tells us that the standard matrix for $T_B \circ T_A$ is BA. That is,

$$T_B \circ T_A = T_{BA}$$

COMPOSITIONS OF THREE OR MORE MATRIX TRANSFORMATIONS

Compositions can be defined analogously for three or more matrix transformations.

$$(T_3 \circ T_2 \circ T_1)(\mathbf{x}) = T_3(T_2(T_1(\mathbf{x})))$$

Or,

$$T_C \circ T_B \circ T_A = T_{CBA}$$

ONE-TO-ONE TRANSFORMATIONS

A matrix transformation T_A: $R^n \to R^m$ is said to be **one-to-one** if T_A maps distinct vectors (points) in R^n to distinct vectors (points) in R^m.

THREE EQUIVALENT STATEMENTS

Theorem 4.3.1: If A is an n n matrix and T_A: $R^n \to R^n$ is the corresponding matrix operator, then the following statements are equivalent.

(a) A is invertible.

(b) The range of T_A is R^n.

(c) T_A is one-to-one.

NOTE: This extends our "big theorem."

INVERSE OF A 1-1 MATRIX OPERATOR

If T_A: $R^n \to R^n$ is a one-to-one matrix operator, then $[T] = A$ is invertible.

The matrix operator $T_{A^{-1}} : R^n \to R^n$ is called the **inverse operator of T_A**.

Notation: The inverse of T is denoted by T^{-1} and its standard matrix is $[T^{-1}] = [T]^{-1}$

PROPERTIES OF MATRIX TRANSFORMATIONS

Theorem 4.10.2: T: $R^n \to R^m$ is a matrix transformation if and and only if the following relationships hold for all vectors **u** and **v** in R^n and every scalar k.

(a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

(b) $T(k\mathbf{u}) = k\,T(\mathbf{u})$ **[Homogeneity property]**

LINEAR TRANSFORMS

The additivity and homogeneity properties in Theorem 4.10.2 are called **linearity conditions**, and a transformation that satisfies these conditions is called a **linear transformation**.

RELATIONSHIP BETWEEN LINEAR AND MATRIX TRANSFORMS

Theorem 4.10.3: Every linear transformation from R^n to R^m is a matrix transformation, and conversely, ever matrix transformation from R^n to R^m is a linear transformation.

THE "BIG" THEOREM

Theorem 4.10.4: If A is an n n matrix, then the following are equivalent.

(a) A is invertible

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row-echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every n 1 matrix **b**.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every n 1 matrix **b**.

(g) $\det(A) \neq 0$

(h) The column vectors of A are linearly independent.

(i) The row vectors of A are linearly independent.

THE "BIG" THEOREM (CONCLUDED)

(j) The column vectors of A span R^n.

(k) The row vectors of A span R^n.

(l) The column vectors of A form a basis for R^n.

(m) The row vectors of A form a basis for R^n.

(n) A has rank n.

(o) A has nullity 0.

(p) The orthogonal complement of the null space of A is R^n.

(q) The orthogonal complement of the row space of A is $[\mathbf{0}]$.

(r) The range of T_A is R^n.

(s) T_A is one-to-one.