# Section 4.1

## Real Vector Spaces

## DEFINITION OF A VECTOR SPACE

Let *V* be any non-empty set of objects on which two operations are defined: addition and multiplication by scalars (numbers).

The operation called **addition** is a rule that associates with each pair of objects **u** and **v** in *V* an object **u** + **v**, called the **sum** of **u** and **v**.

The operation called **scalar multiplication** is a rule that associates with each scalar *k* and each object **u** in *V* an object *k***u**, called the **scalar multiple** of **u** by *k*.

If the following ten axioms are satisfied by all objects, **u**, **v**, **w** in *V* and all scalars *k* and *m*, then we call *V* a **vector space** and the objects in *V* **vectors**.

If the scalars are ***real*** numbers, we call *V* a **real vector space**.

If the scalars are ***complex*** numbers, we call *V* a **complex vector space**.

## THE TEN VECTOR SPACE AXIOMS

1. If **u** and **v** are objects in *V*, then **u** + **v** is in *V*.
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. There is an object **0** in *V*, called a **zero vector** for *V*, such that **0** + **u** = **u** + **0** = **u** for all **u** in *V*.
5. For each **u** in *V*, there is an object –**u** in *V*, called a **negative** of **u**, such that **u** + (–**u**) = (–**u**) + **u** = **0**.
6. If *k* is any scalar and **u** in any object in *V*, then *k***u** is in *V*.
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. 1**u** = **u**

## COMMENT ON VECTOR SPACE AXIOMS

The vector space axioms are divided into two parts.

- Axioms 1 through 5 concern vector addition.
- Axioms 6 through 10 concern scalar multiplication.

## HOW TO SHOW A SET WITH TWO OPERATIONS IS A VECTOR SPACE

**Step 1:** Identify the set *V* of objects that will become vectors

**Step 2:** Identify the addition and scalar multiplication operations on *V*.

**Step 3:** Verify Axioms 1 and 6: that is, adding two vectors in *V* produces a vector in *V*, and multiplying a vector in *V* by a scalar produces a vector in *V*. Axiom 1 is called **closure under addition**, and Axiom 6 is called **closure under scalar multiplication**.

**Step 4:** Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

## EXAMPLES OF VECTOR SPACES

- $R^n$ together with standard vector addition and standard scalar multiplication
- $M_{2\ 2}$, the set of 2  2 matrices with standard matrix addition and scalar multiplication
- $F(-\infty, \infty)$, the set of all real-valued functions having domain $(-\infty, \infty)$ with standard addition and scalar multiplication

## EXAMPLES (CONCLUDED)

- $P_2$, the set of all polynomials of degree at most 2, with standard polynomial addition and scalar multiplication
- $V = \{(v_1, v_2) | v_1, v_2 > 0\}$ with $(u_1, u_2) + (v_1, v_2) = (u_1v_1, u_2v_2)$ and $k(u_1, u_2) = (u_1^k, u_2^k)$
- The zero vector space.

## EXAMPLES THAT ARE ***NOT*** VECTOR SPACES

- $V$ = set of ordered triples with standard addition and scalar multiplication defined by $k(x, y, z) = (kx, y, z)$
- $V = \{(x, y) \mid x \geq 0\}$ with standard addition and standard scalar multiplication

## SOME PROPERTIES OF VECTORS

**Theorem 4.1.1:** Let $V$ be a vector space, $\mathbf{u}$ a vector in $V$, and $k$ a scalar; then:

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{u} = -\mathbf{u}$

(d) If $k\mathbf{u} = \mathbf{0}$, then either $k = 0$ or $\mathbf{u} = \mathbf{0}$.

## A CLOSING OBSERVATION

"This section of the [course] is very important to the overall plan of linear algebra in that it establishes a common thread between such diverse mathematical objects as geometric vectors, vectors in $R^n$, infinite sequences, matrices, and real-valued functions, to name a few. As a result, whenever we discover a new theorem [or property] about general vector spaces, we will at the same time be discovering a new theorem [or property] about geometric vectors, vectors in $R^n$, infinite sequences, matrices, real-valued functions, and about any new kinds of vectors that we might discover."