Section 3.3

Orthogonality

ORTHOGONAL VECTORS

Two nonzero vectors **u** and **v** in R^n are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n. A nonempty set of vectors in R^n is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an **orthonormal set**.

VECTORS, LINES, AND PLANES

Theorem 3.3.1:

(a) If a and b are constants that are not both zero, then an equation of the form

$$ax + by + c = 0$$

represents a line in R^2 with normal $\mathbf{n} = (a, b)$.

(b) If a, b, and c are constants that are not all zero, then an equation of the form

$$ax + by + cz + d = 0$$

represents a plane in R^3 with normal $\mathbf{n} = (a, b, c)$.

THE PROJECTION THEOREM

Theorem 3.3.2 Projection Theorem:

If **u** and **a** are vectors in R^n, and if $\mathbf{a} \neq \mathbf{0}$, then **u** can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of **a** and $\mathbf{w}_2$ is orthogonal to **a**.

ORTHOGONAL PROJECTION

The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ from the Prejection Theorem have associated names. The vector $\mathbf{w}_1$ is called the **orthogonal projection of u on a** or sometimes the **vector component of u along a**. It is denoted by

$$\text{proj}_{\mathbf{a}}\, \mathbf{u}.$$

The vector $\mathbf{w}_2$ is called the **vector component of u orthogonal to a** and can be written as

$$\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\, \mathbf{u}.$$

SUMMARY OF PROJECTIONS

$$\text{proj}_{\mathbf{a}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$ [vector component of u along a]

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\, \mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$ [vector component of u orthogonal to a]

NORM OF THE ORTHOGONAL PROJECTION OF u ALONG a

$$\|\text{proj}_{\mathbf{a}}\,\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{a}|}{\|\mathbf{a}\|}$$

and

$$\|\text{proj}_{\mathbf{a}}\,\mathbf{u}\| = \|\mathbf{u}\|\cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{a}$

PYTHAGOREAN THEOREM IN R^n

Theorem 3.3.3: If **u** and **v** are ***orthogonal*** vectors in R^n with the Euclidean inner product, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

DISTANCE BETWEEN A POINTS AND LINES AND PLANES

Theorem 3.3.4:

(a) In R^2, the distance D between the point $P(x_0, y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

(b) In R^2, the distance D between the point $P(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$