Section 3.2

Norm, Dot Product, and Distance in *R*^{n}

NORM OF A VECTOR

The length of a vector **v** is often called the **norm** of **v** and is denoted by ||**v**||. By the Pythagorean Theorem, we have

$$\| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2} \text{ if } \mathbf{v} = (v_1, v_2) \text{ in } 2-\text{space}$$

$$\| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2 + v_3^2} \text{ if } \mathbf{v} = (v_1, v_2, v_3) \text{ in } 3-\text{space}$$

NORM OF A VECTOR IN *R*^{n}

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the **norm** of **v** (also called the **length** of **v** or the **magnitude** of **v**) is denoted by ||**v**||, and is defined by the formula

$$\| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

THEOREM 3.2.1

Theorem 3.2.1: If **v** is a vector in R^n, and if k is any scalar, then:

(a) $\|\mathbf{v}\| \geq 0$

(b) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

(c) $\|k\mathbf{v}\| = |k| \, \|\mathbf{v}\|$

UNIT VECTORS

A vector of length 1 is called a **unit vector**.

If **v** is any nonzero vector in R^n, then

$$\mathbf{u} = \frac{1}{\| \mathbf{v} \|} \mathbf{v} = \frac{\mathbf{v}}{\| \mathbf{v} \|}$$

defines a unit vector that is in the same direction as **v**.

STANDARD UNIT VECTORS

When a rectangular coordinate system is introduced in R^2 and R^3, the unit vectors in the positive directions of the coordinate axes are called the **standard unit vectors**. In R^2, these vectors are denoted by

$$\mathbf{i} = (1, 0) \quad \text{and} \quad \mathbf{j} = (0, 1)$$

and in R^3 by

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \text{and} \quad \mathbf{k} = (0, 0, 1)$$

STANDARD UNIT VECTORS (CONCLUDED)

In R^n, the standard unit vectors in R^n are defined to be

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),$$

$$\ldots,\quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

in which case every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in Rn can be expressed as

$$\mathbf{v} = (v_1, v_2, \ldots, v_n) =$$

$$v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots + v_n\mathbf{e}_n$$

DISTANCE BETWEEN VECTORS

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in Rn, then we denote the distance between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{(u_2-v_1)^2+(u_2-v_1)^2+\cdots+(u_n-v_n)^2}$$

THE DOT PRODUCT IN R^2 AND R^3

If $\mathbf{u}$ and $\mathbf{v}$ are nonzerp vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the **dot product** (also called the **Euclidean inner product**) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

THE DOT PRODUCT IN R^n

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are any vectors in R^n, then the **dot product** (also called the **Euclidean inner product**) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n.$$

THE DOT PRODUCT AND NORM

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$$

PROPERTIES OF THE DOT PRODUCT

Theorem 3.2.2: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, and if k is any scalar, then:

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **[Symmetric property]**

(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **[Distributive property]**

(c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$ **[Homogeneity property]**

(d) $\mathbf{v} \cdot \mathbf{v} > 0$ if $\mathbf{v} \neq \mathbf{0}$, **[Positivity property]**

and $\mathbf{v} \cdot \mathbf{v} = 0$ if $\mathbf{v} = \mathbf{0}$

MORE PROPERTIES OF THE DOT PRODUCT

Theorem 3.2.3: If **u**, **v**, and **w** are vectors in R^n, and if k is any scalar, then:

(a) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(c) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(d) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(e) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

THE ANGLE BETWEEN VECTORS IN R^n

We extend the idea of the angle between vectors to R^n by defining the angle θ between vectors **u** and **v** with the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}\right)$$

CAUCHY-SCHWARZ INEQUALITY IN R^n

Theorem 3.2.4: If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

THE TRIANGLE INEQUALITIES

Theorem 3.2.5: If **u**, **v**, and **w** are vectors in R^n, then:

(a) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$
[Triangle inequality for vectors]

(b) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$
[Triangle inequality for distance]

PARALLELOGRAM EQUATION FOR VECTORS

Theorem 3.2.6 Parallelogram Equation for Vectors: If **u** and **v** are vectors in R^n, then

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right)$$

A THEOREM

Theorem 3.2.7: If **u** and **v** are vectors in R^n with the Euclidean inner product, then

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$$