Section 3.1

Vectors in 2-Space, 3-Space, and *n*-Space

SCALARS AND VECTORS

A **scalar** is a quantity that is completely described by it magnitude alone.

Examples: area, length, mass, temperature, speed.

A **vector** is a quantity that is not completely determined until both a magnitude and a direction are specified.

Examples: velocity, force, displacement, acceleration.

GEOMETRIC VECTORS

Geometric vectors have length and direction. Vectors can be represented as arrows in two dimensions (also called **2-space**) or in three dimensions (also called **3-space**). The direction of the arrowhead specifies the **direction** of the vector and the **length** of the arrow specifies the magnitude. The tail of the arrow is called the **initial point** of the vector, and the tip of the arrow is the **terminal point**. We will denote vectors in lower case boldface type (**a**, **u**, **v**, etc.).

The length of a vector is called its **magnitude**.

EQUIVALENT VECTORS AND EQUAL VECTORS

Vectors with the same length and direction are called **equivalent**.

Since we want a vector to be determined solely by its length and direction, equivalent vectors are regarded as **equal** even though they may be located in different positions.

If **v** and **w** are equivalent, we write **v** = **w**.

PARALLELOGRAM RULE FOR VECTOR ADDITION

If **v** and **w** are two vectors in 2-space or 3-space that are positioned so that their initial points coincide, then the two vectors form adjacent sides of a parallelogram, and the **sum** **v** + **w** is the vector represented by the arrow from the common initial point of **v** and **w** to the opposite vertex of the parallelogram.

TRIANGLE RULE FOR VECTOR ADDITION

If **v** and **w** are two vectors in 2-space or 3-space that are positioned so that the initial point of **w** is at the terminal point of **v**, then the **sum** **v** + **w** is the vector represented by the arrow from the initial point of **v** to the terminal point of **w**.

ZERO VECTOR; NEGATIVE VECTORS

The **zero vector** **0** is the vector of length zero.

The **negative of** **v**, denoted by −**v**, is the vector having the same magnitude as **v**, but pointing in the opposite direction.

DIFFERENCE OF VECTORS

If **v** and **w** are any two vectors, then the **difference** of **w** from **v** is defined by

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\,\mathbf{w}).$$

SCALAR MULTIPLICATION

If **v** is a nonzero vector in 2-space or 3-space, and if k is a nonzero scalar, then we define the **scalar product of v by *k*** (denoted by $k\mathbf{v}$) to be the vector whose length is $|k|$ times the length of **v** and whose direction is the same as that of **v** if k is positive and opposite to that of **v** if k is negative. We define $k\mathbf{v} = \mathbf{0}$ if $k = 0$ or $\mathbf{v} = \mathbf{0}$.

A vector of the form $k\mathbf{v}$ is called a **scalar multiple** of **v**.

ASSOCIATIVE LAW OF ADDITION

If **u**, **v**, and **w** are vectors in 2-space or 3-space, then

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

VECTORS IN COORDINATE SYSTEMS

If a vector **v** in 2-space or 3-space is positioned with its initial point at the origin of the rectangular coordinate system, then the vector is completely determined by the coordinates of its terminal point. We call these coordinates the **components** of **v** relative to the coordinate system.

We will write $\mathbf{v} = (v_1, v_2)$ to denote a vector **v** in 2-space with components (v_1, v_2), and $\mathbf{v} = (v_1, v_2, v_3)$ to denote a vector in 3-space with components (v_1, v_2, v_3).

EQUIVALENT VECTORS

In terms of components, two vectors are equivalent if and only if their corresponding components are equal. Thus, in 3-space, the vectors

$$\mathbf{v} = (v_1, v_2, v_3) \quad \text{and} \quad \mathbf{w} = (w_1, w_2, w_3)$$

are equivalent if and only if

$$v_1 = w_1, \quad v_2 = w_2, \quad v_3 = w_3$$

FINDING THE COMPONENT FORM OF A VECTOR

If a vector **v** in 2-space has initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the component form of **v** is

$$\mathbf{v} = \overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1)$$

If a vector **v** in 3-space has initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$, then the component form of **v** is

$$\mathbf{v} = \overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1, z_2 - z_1)$$

n-SPACE

If n is a positive integer, then an **ordered *n*-tuple** is a sequence of n real numbers $(a_1, a_2, a_3, \ldots, a_n)$.

The set of all ordered n-tuples is called ***n*-space** and is denoted by R^n.

EQUIVALENT VECTORS

Vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in R^n are said to be **equivalent** (also called **equal**) if

$$v_1 = w_1, \quad v_2 = w_2, \ldots, \quad v_n = w_n$$

We indicate this by writing $\mathbf{v} = \mathbf{w}$.

ADDITION, SUBTRACTION, SCALAR MULTIPLICATION IN COMPONENT FORM

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n and if k is any scalar, then

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1,\ v_2 + w_2,\ \ldots, v_n + w_n)$$
$$k\mathbf{v} = (kv_1,\ kv_2,\ \ldots, kv_n)$$
$$-\mathbf{v} = (-v_1,\ -v_2,\ \ldots, -v_n)$$
$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = (v_1 - w_1,\ v_2 - w_2,\ \ldots, v_n - w_n)$$

PROPERTIES OF VECTORS IN R^n

Theorem 3.1.1: If **u**, **v**, and **w** are vectors in R^n and k and m are scalars, then:

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$; *i.e.*, $\mathbf{u} - \mathbf{u} = \mathbf{0}$

(e) $k(m\mathbf{u}) = (km)\mathbf{u}$

(f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(g) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$

THEOREM 3.1.2

Theorem 3.1.2: If **v** is a vector in R^n and k is a scalar, then:

(a) $0\mathbf{v} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{v} = -\mathbf{v}$

LINEAR COMBINATION

If **w** is a vector in R^n, then **w** is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the **coefficients** of the linear combination. In the case where $r = 1$, the formula becomes $\mathbf{w} = k_1\mathbf{v}_1$, so that a linear combination of a single vector is just scalar multiplication of that vector.