# Section 2.3

**Properties of Determinants;**
**Cramer's Rule**

## det(*kA*)

Let $A$ be an $n \times n$ matrix. Then

$$\det(kA) = k^n \det(A)$$

## A THEOREM

**<u>Theorem 2.3.1</u>:** Let $A$, $B$, and $C$ be $n \times n$ matrices that differ only in a single row, say the $r^{\text{th}}$, and assume that the $r^{\text{th}}$ row of $C$ can be obtained by adding corresponding entries in the $r^{\text{th}}$ rows of $A$ and $B$. Then

$$\det(C) = \det(A) + \det(B)$$

The same result holds for columns.

## A LEMMA

**<u>Lemma 2.3.2</u>:** If $B$ is an $n \times n$ matrix and $E$ is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B)$$

This lemma generalizes as follows:

$$\det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)$$

## INVERTIBILITY AND DETERMINANTS

**<u>Theorem 2.3.3</u>:** A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.

## MATRIX MULTIPLICATION AND DETERMINANTS

**<u>Theorem 2.3.4</u>:** If $A$ and $B$ are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B)$$

## DETERMINANT OF THE INVERSE

**Theorem 2.3.5:** If $A$ is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

## ADJOINT OF A MATRIX

If $A$ is any $n \times n$ matrix and $C_{ij}$ is the cofactor of $a_{ij}$, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the **matrix of cofactors from $A$**. The transpose of this matrix is called the **adjoint of $A$** and is denoted by adj($A$).

## INVERSE OF A MATRIX USING ITS ADJOINT

**Theorem 2.3.6:** If $A$ is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

## CRAMER'S RULE

**Theorem 2.3.7: Cramer's Rule**

If $A\mathbf{x} = \mathbf{b}$ is a system of linear equations in $n$ unknown such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \cdots,\ x_n = \frac{\det(A_n)}{\det(A)}$$

where $A_j$ is the matrix obtained by replacing the entries in the $j^{\text{th}}$ column of $A$ by the entries in the matrix $\mathbf{b}$.

## EQUIVALENT STATEMENTS

**Theorem 2.3.8:** If $A$ is an $n \times n$ matrix, then the following statements are equivalent; that is, all are true or all are false.

(a) $A$ is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row-echelon form of $A$ is $In$.

(d) $A$ is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(g) $\det(A) \neq 0$