Section 2.1

Determinants by Cofactor Expansion

THE DETERMINANT

Recall from algebra, that the function $f(x) = x^2$ is a function from the real numbers into the real numbers.

In a similar way, the **determinant** is a function from square matrices into the real numbers; that is, the input is a square matrix and the output is a real number.

Notation: The determinant of the matrix A is notated by $\det(A)$ or $|A|$.

DETERMINANT OF A 2×2 MATRIX

Recall that the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if $ad - bc \neq 0$. The determinant of the matrix A is

$$\det(A) = |A| = ad - bc.$$

Thus,

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

MINORS AND COFACTORS

If A is a square matrix, then the **minor of entry a_{ij}** is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i^{th} row and j^{th} column are deleted from A. The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} and is called the **cofactor of entry a_{ij}**.

THEOREM 2.1.1

Theorem 2.1.1: If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

COFACTOR EXPANSION

If A is any an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the **determinant of A**, and the sums themselves are called **cofactor expansions of A**. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

A COMMENT ON COFACTOR EXPANSION

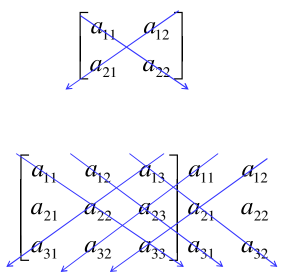
When computing the determinant by cofactor expansion, it is helpful to expand along the row or column that contains the most zeros.

DETERMINANTS AND TRIANGULAR MATRICES

Theorem 2.1.2: If A is an $n\times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is

$$\det(A) = a_{11}\, a_{22} \cdots a_{nn}$$

A USEFUL TECHNIQUE FOR EVALUATING 2×2 AND 3×3 DETERMINANTS



WARNING: The arrow technique only works for 2×2 and 3×3 matrices.