Section 1.6

More on Linear Systems and Invertible Matrices

SOLUTIONS TO LINEAR SYSTEMS

Theorem 1.6.1: Every system of linear equations has either

- no solutions,
- exactly one solution, or
- infinitely many solutions.

SOLVING A SYSTEM BY MATRIX INVERSION

Theorem 1.6.2: If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations

$$A\mathbf{x} = \mathbf{b}$$

has exactly one solution, namely,

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

SQUARE MATRICES AND INVERSES

Theorem 1.6.3: Let A be a square matrix.

(a) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.

(b) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

EQUIVALENT STATEMENTS

Theorem 1.6.4: If A is an $n \times n$ matrix, then the following statements are equivalent; that is, all are true or all are false.

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row-echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

INVERTIBILITY OF A PRODUCT

Theorem 1.6.5: Let A and B be square matrices of the same size. If the product AB is invertible, then A and B must also be invertible.