Section 1.4

Inverses; Rules of Matrix Arithmetic

THEOREM 1.4.1: PROPERTIES OF MATRIX ARITHMETIC

(a) $A + B = B + A$ (**Commutative law for addition**)

(b) $A + (B + C) = (A + B) + C$ (**Associative law for add.**)

(c) $A(BC) = (AB)C$ (**Associative law for multiplication**)

(d) $A(B + C) = AB + AC$ (**Left distributive law**)

(e) $(A + B)C = AC + BC$ (**Right distributive law**)

(f) $A(B - C) = AB - AC$

(g) $(A - B)C = AC - BC$

(h) $a(B + C) = aB + aC$

THEOREM 1.4.1 (CONTINUED)

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

Note: Since multiplication is ***not*** commutative, we need ***two*** distributive laws: the left distributive law and the right distributive law.

ZERO MATRICES

A **zero matrix** is a matrix that has zeros for all its entries. The zero matrix is denoted by 0.

Theorem 1.4.2: Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be performed, then:

(a) $A + 0 = 0 + A = A$

(b) $A - A = 0$

(c) $0 - A = -A$

(d) $A0 = 0$; $0A = 0$

IDENTITY MATRICES

A square matrix that has 1's on the main diagonal and 0's off the main diagonal is called an **identity matrix**. The identity matrix is denoted by I.

Example: A 3 × 3 identity matrix. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note: An identity matrix has the property that

$$AI = IA = A.$$

THEOREM

Theorem 1.4.3: If R is the reduced row-echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n.

INVERSE OF A MATRIX

If A is a square matrix, and if a matrix B of the same size can be found such that

$$AB = BA = I,$$

then A is said to be **invertible** (or **nonsingular**) and B is called an **inverse** of A.

If no such matrix B can be found, then A is said to be **singular**.

UNIQUENESS OF THE INVERSE

Theorem 1.4.4: If B and C are both inverses of A, then $B = C$.

Notation: For an invertible matrix A, we write its inverse as A^{-1}.

INVERSE OF A 2×2 MATRIX

Theorem 1.4.5: The matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

INVERTIBILITY OF A PRODUCT

Theorem 1.4.6: If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Generalization: A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

EXPONENTS

If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \qquad A^n = \underbrace{AA\cdots A}_{n \text{ factors}} \quad (n > 0)$$

Moreover, if A is invertible, then we define the negative powers to be

$$A^{-n} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ factors}}$$

TWO LAWS OF EXPONENTS

If A is a square matrix and r and s are integers, then

$$A^rA^s = A^{r+s}, \qquad (A^r)^s = A^{rs}$$

LAWS OF EXPONENTS PART 2

Theorem 1.4.7: If A is an invertible matrix and n is a nonnegative integer, then:

(a) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(b) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$ for $n = 1, 2, \ldots$.

(c) For any nonzero scalar k, the matrix kA is invertible and $(kA)^{-1} = k^{-1}A^{-1} = \frac{1}{k}A^{-1}$.

PROPERTIES OF THE TRANSPOSE

Theorem 1.4.8: If the sizes of the matrices are such that the stated operations can be performed, then

(a) $((A)^T)^T = A$

(b) $(A+B)^T = A^T + B^T$ and $(A-B)^T = A^T - B^T$

(c) $(kA)^T = k(A^T)$ where k is any scalar

(d) $(AB)^T = B^T A^T$

A GENERALIZATION

The transpose of any number of matrices is equal to the product of the transposes in the reverse order.

INVERTIBILITY AND TRANSPOSES

Theorem 1.4.9: If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$