Section 1.2

Gaussian Elimination

REDUCED ROW-ECHELON FORM

A matrix is in **reduced row-echelon form** if it has the following properties.

1. If a row does not consist of all zeros, the first nonzero number must be a 1 (called a **leading 1**).
2. Any rows consisting of all zeros are grouped together at the bottom of the matrix.
3. In any two successive nonzero rows, the leading 1 in lower row occurs farther to the right than the leading 1 in the higher row.
4. Each ***column*** that has a leading 1 has zeros everywhere else.

ROW-ECHELON FORM

A matrix that has only properties 1, 2, and 3 is said to be in **row-echelon form**.

EXAMPLES

Reduced Row-Echelon Form:

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

Row-Echelon Form:

$$\begin{bmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

ELIMINATION METHODS

- **Gaussian elimination** uses row operations to produce a ***row-echelon*** matrix.
- **Gauss-Jordan elimination** uses row operations to produce a ***reduced row-echelon*** matrix.

GENERAL SOLUTION TO A SYSTEM

If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is call a **general solution** of the system.

HOMOGENEOUS SYSTEMS

A system of linear equations is **homogeneous** if the constant terms are all zero.

Example:

$$\begin{aligned} x_1 - 2x_2 + x_3 + 4x_4 &= 0 \\ 3x_1 - 6x_2 + 12x_4 &= 0 \\ 3x_1 - 6x_2 + 2x_3 + 12x_4 &= 0 \\ -2x_1 + 4x_2 + x_3 - 8x_4 &= 0 \end{aligned}$$

NOTES ON HOMOGENEOUS SYSTEMS

- ***All*** homogeneous systems are consistent since
  $$x_1 = 0, x_2 = 0, \ldots, x_n = 0$$
  is a solution.
- The solution above is called the **trivial solution**.
- Any other solution to a homogenous system is a **nontrivial solution**.

TWO THEOREMS

Theorem 1.2.1 Free Variable Theorem for Homogeneous Systems: If a homogeneous system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem 1.2.2: A homogenous system of linear equations with more unknowns than equations has infinitely many solutions.