Section 5.1

Polynomial Functions

POLYNOMIAL FUNCTIONS

A **polynomial function** in one variable is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is the variable. If $a_n \neq 0$, it is called the **leading coefficient**, and n is the **degree** of the polynomial.

The domain of a polynomial function is all real numbers.

SOME POLYNOMIAL FUNCTIONS

1. $f(x) = -4x^5 - 7x^4 + 3x^2 - x + 1$
2. $f(x) = 2x + 1$
3. $f(x) = -6$
4. $f(x) = x^3 + \sqrt{7}x^2 - \frac{1}{2}x$

SOME "NON-POLYNOMIAL" FUNCTIONS

$f(x) = \frac{x^3}{x^2-3x-4}$ (a **rational function**)

$f(x) = \sqrt[3]{x^2 + x + 1}$ (an **algebraic function**)

HOW TO FIND THE DEGREE OF A POLYNOMIAL WRITTEN IN FACTORED FORM

To determine the degree of a polynomial written in factored form, add the powers of all the factors.

EXAMPLE: $f(x) = 5(x-2)^3(x+2)^2(x+4)$

TERMINOLOGY ASSOCIATED WITH POLYNOMIALS

- A **monomial** is a polynomial with only one term. For example: $f(x) = 3x^4$.
- The monomials that make up a polynomial are called its **terms**.
- If $a_n \neq 0$, ax^n is called the **leading term**.
- The term a_0 is called the **constant term**.
- If all the coefficients of the polynomial are 0, it is called the **zero polynomial**, which has no degree.
- Polynomials are usually written in **standard form**, beginning with the nonzero terms of highest degree and continuing with terms in decreasing order according to degree.
- A **binomial** is a polynomial with exactly two terms. For example: $f(x) = 4x - 3$.
- A **trinomial** is a polynomial with exactly three terms. For example: $f(x) = 3x^3 - 2x - 1$.

POLYNOMIALS PREVIOUSLY STUDIED

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero Function	The x-axis
0	$f(x) = a_0$, $a_0 \neq 0$	Constant Function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0$, $a_1 \neq 0$	Linear Function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0$, $a_2 \neq 0$	Quadratic Function	Parabola: opens up if $a_2 > 0$; opens down if $a_2 < 0$

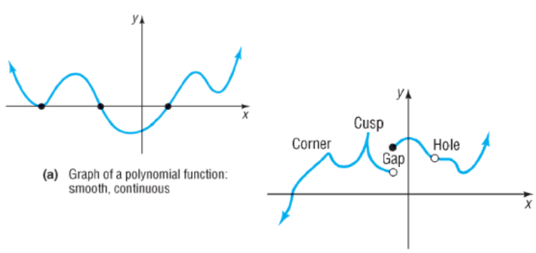
SMOOTH CONTINUOUS CURVES

A **smooth curve** is a curve (graph) that does not have sharp corners.

A **continuous curve** is a curve that does not have a break or hole in it.

A smooth and continuous curve is a curve that does not have sharp corners, breaks, or holes.

All polynomials are smooth and continuous.



(a) Graph of a polynomial function: smooth, continuous

(b) Cannot be the graph of a polynomial function

POWER FUNCTION

A **power function of degree *n*** is a monomial of the form

$$f(x) = ax^n$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

PROPERTIES OF POWER FUNCTIONS OF EVEN DEGREE

Let $f(x) = x^n$ be a power function where n is a positive even integer.

1. The graph of the function f is symmetric with respect to the y-axis.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(-1, 1)$, $(0, 0)$, and $(1,1)$.
4. As the exponent increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

PROPERTIES OF POWER FUNCTIONS OF ODD DEGREE

Let $f(x) = x^n$ be a power function where n is a positive odd integer.

1. The graph of the function f is symmetric with respect to the origin.
2. The domain is the set of all real numbers. The range is the set of all real numbers.
3. The graph always contains the points $(-1, -1)$, $(0, 0)$, and $(1,1)$.
4. As the exponent increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

REAL ZEROS OF A POLYNOMIAL

If f is a function and r is a real number for which $f(r) = 0$, then r is called a **real zero** of f. It is also referred to as a **real root** of f.

In particular, if f is a polynomial function, the following are equivalent:

1. r is a real zero of a polynomial function f.
2. r is an x-intercept of the graph of f.
3. $x - r$ is a factor of f.
4. r is a solution to the equation $f(x) = 0$.

MULTIPLE ZEROS

If the same factor $x - r$ occurs more than once, r is called a **repeated**, or **multiple**, **zero** of f.

Another way to say this is:

If $(x - r)^m$ is a factor of the polynomial f and $(x - r)^{m+1}$ is NOT a factor of f, then r is called a ***zero of multiplicity m of f***.

ZEROS OF EVEN AND ODD MULTIPLICITY

If r is a zero of even multiplicity

Numerically: The sign of $f(x)$ does not change from one side to the other of r.

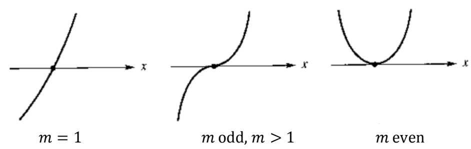
Graphically: The graph of f **touches** the x-axis at r.

If r is a zero of odd multiplicity

Numerically: The sign of $f(x)$ changes from one side to the other of r.

Graphically: The graph of f **crosses** the x-axis at r.

ROOTS OF MULTIPLICITY *m*



LOCAL MAXIMA AND LOCAL MINIMA

- A y-value is a **local maximum** if it is the largest y-value anywhere in its neighborhood on the graph.
- A y-value is a **local minimum** if it is the smallest y-value anywhere in its neighborhood on the graph.

TURNING POINTS

The points at which a graph changes direction are called **turning points**. Each turning points yields a local maximum or local minimum.

TURNING POINTS AND DEGREE

- If f is a polynomial function of degree n, then the graph of f has at most $n - 1$ turning points.
- If the graph of a polynomial function f has $n - 1$ turning points, then the degree of f is at least n.

END BEHAVIOR

The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its **end behavior**.

END BEHAVIOR OF POLYNOMIALS

For large values of x, either positive or negative, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0$$

resembles the graph of the power function

$$y = a_n x^n$$

SUMMARY OF END BEHAVIOR

	Exponent of Leading Term Even	Exponent of Leading Term Odd
Leading Term Positive	Up to left and up to right	Down to left and up to right
Leading Term Negative	Down to left and down to right	Up to left and down to right

GRAPH OF A POLYNOMIAL FUNCTION

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_n \neq 0$ be a polynomial function.

- Degree of f: n
- y-intercept: $(0, a_0)$
- The graph is smooth and continuous.
- Maximum number of turning points: $n - 1$
- At a zero of even multiplicity: The graph touches the x-axis.
- At a zero of odd multiplicity: The graph crosses the x-axis.
- Between the zeros, the graph of f is either above or below the x-axis.
- End behavior: For large $|x|$, the graph of f behaves like the graph of $y = a_n x^n$.

ANALYZING THE GRAPH OF A POLYNOMIAL FUNCTION

Step 1: Determine the end behavior of the graph of the function.

Step 2: Find the x- and y-intercepts of the graph of the function.

Step 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

Step 4: Determine the maximum number of turning points of the graph of the function.

Step 5: Use the information in Steps 1 through 4 to draw a complete graph of the function. To help establish the y-axis scale, find additional points on the graph on each side of any x-intercepts.