Section 4.2

The Definite Integral

THE DEFINITE INTEGRAL

If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $a = x_0, x_1, x_2, x_3, \ldots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so that x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

REMARKS

The symbol $\int$ is called an **integral sign**.

In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand**, and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**.

The symbol dx has no official meaning by itself; $\int_a^b f(x)dx$ is all one symbol.

The procedure of calculating an integral is called **integration**.

THE RIEMANN SUM

The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

from the definition of the definite integral is called a **Riemann sum** after the German mathematician Bernhard Riemann.

FUNCTIONS THAT ARE INTEGRABLE

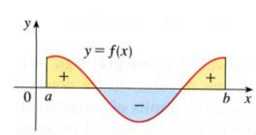
Theorem: If f is a continuous function on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

NET AREA

If $f(x) \ge 0$ (that is, the graph lies above the x-axis), $\int_a^b f(x)dx$ gives the area under the curve.

If the graph of $f(x)$ lies both above and below the x-axis, then the definite integral gives the **net area** (the area above the x-axis subtracted by the area below the x-axis); that is

$$\int_a^b f(x)dx = A_{\text{up}} - A_{\text{down}}$$



THE MIDPOINT RULE

If we are approximating a definite integral, it is often better to let x_i^* be the midpoint of the ith subinterval. This results in the **Midpoint Rule**.

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where $\Delta x = \frac{b-a}{n}$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) =$ midpoint of $[x_{i-1}, x_i]$

BASIC PROPERTIES OF THE DEFINITE INTEGRAL

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

CONSTANT MULTIPLE AND ADDITION/SUBTRACTION PROPERTIES

1. $\int_a^b c\,dx = c(b-a)$
2. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
3. $\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$
4. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

INTERVAL ADDITIVITY PROPERTY

5. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

COMPARISON PROPERTIES

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$