Section 2.2

The Derivative as a Function

THE DERIVATIVE AS A FUNCTION

Given any number x for which the limit exists, we define a new function $f'(x)$, called the **derivative of *f***, by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

NOTE: The function f' can be graphed and studied just like any other function.

OTHER NOTATIONS FOR THE DERIVATIVE

Below are other notations for the derivative function, $f'(x)$. All of the these notation are used interchangeably.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D\,f(x) = D_x f(x)$$

The symbols D and $\frac{dy}{dx}$ are called **differential operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

LEIBNIZ NOTATION

The $\frac{dy}{dx}$ notation is called **Leibniz notation**. This notation comes from the increment notation; that is,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If we want to indicate the value of $\frac{dy}{dx}$ at the value a using Leibniz notation, we write

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$. The vertical bar means "evaluate at."

DIFFERENTIABILITY

Definition: A function f is **differentiable at *a*** if $f'(a)$ exists. The function f is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, b)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

DIFFERENTIABILITY AND CONTINUITY

Theorem: If f is differentiable at a, then f is continuous at a.

NOTE: The converse of this theorem is false; that is, there are functions that are continuous at a point but not differentiable at that point.

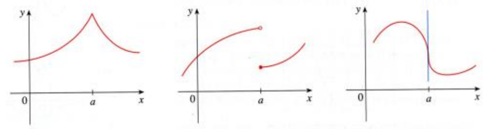
HOW CAN A FUNCTION FAIL TO BE DIFFERENTIABLE?

A function, f, can fail to be differentiable in three ways.

(a) The graph can have a corner (or sharp point).

(b) The graph can have a discontinuity.

(c) The graph can have a vertical tangent.



(a) A corner (b) A discontinuity (c) A vertical tangent

HIGHER ORDER DERIVATIVES

- The **second derivative** of $y = f(x)$ is

$$y'' = f''(x) = D_x^2 f(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

- The **third derivative** of $y = f(x)$ is

$$y''' = f'''(x) = D_x^3 f(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

- The ***n*th derivative** of $y = f(x)$ is

$$y^{(n)} = f^{(n)}(x) = D_x^n f(x) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}$$