1. Find the differential $dy$ of the following functions.
   (a) $y = 7x^3 - 3x^2 + 4$ 
   (b) $y = \frac{x}{1-x}$ 
   (c) $y = \sec x$

2. Use the information to evaluate and compare $\Delta y$ and $dy$.
   (a) $y = x^2 - 3; \ x = 2; \ dx = \Delta x = 0.5$ 
   (b) $y = \sqrt{25 - x^2}; \ x = 3; \ dx = \Delta x = -0.1$ Round your answer to three decimal places.

3. Determine the linear approximation of the function $f(x) = \sqrt{x^2 + 9}$ at $a = 4$. Use it to approximate $\sqrt{25.81}$.

4. (a) Use differentials to approximate the volume of material used in a spherical shell of inner radius 100 cm and outer radius 100.2 cm.
   (b) All sides of a cubical metal box are 0.5 inches thick and the volume of the interior of the box is 200 cubic inches. Use differentials to approximate the volume of metal used to make the box.
   (c) A metal disk has an area of $100\pi$ square inches. When the disk is heated in an oven to 500°F, the radius of the disk increases by 0.25 inches. Use differentials to approximate the change in the area of metal disk.

5. Determine the absolute extrema (maximum and minimum) for each of the following functions and the $x$-value in the closed interval where it occurs.
   (a) $f(x) = x^3 - 3x + 1; \ [-\frac{3}{2}, 3]$ 
   (b) $f(x) = \frac{x}{x^2 + 2}; \ [-1, 4]$ 
   (c) $g(x) = x^{2/5}; \ [-1, 32]$

6. Determine whether the Mean Value Theorem (for derivative) applies to the given function on the indicated interval. If it does, state why and find all possible numbers $c$ that satisfy the conclusion of the Mean Value Theorem. If the Mean Value Theorem does not apply, state the reason why.
   (a) $f(x) = x^3 - 2x; \ [-1, 2]$ 
   (b) $f(x) = 1 - |x|; \ [-3, 3]$ 
   (c) $f(x) = \sqrt{x}; \ [4, 9]$ 
   (d) $f(x) = x - \sin x; \ [0, \pi]$ 
   (e) $f(x) = \frac{1}{x}; \ [-1, 1]$
7. Use the indicated test to find all local extrema (maxima and minima) for the following functions.
(a) \( f(x) = 2x + x^{2/3} \) Use the \textbf{First Derivative Test}.
(b) \( f(x) = -2x^3 + 3x^2 + 12x + 1 \) Use the \textbf{Second Derivative Test}.
(c) \( f(x) = x + \frac{1}{x} \) Use either test.
(d) \( f(x) = \frac{\sin x}{2 + \cos x}; \quad 0 < x < 2\pi \) Use either test.

8. (a) Determine the intervals on which \( f(x) = -x^3 + 6x^2 - 9x \) is increasing and decreasing.
(b) Determine the intervals on which \( f(x) = x^3 - 6x^2 - 12x - 8 \) is concave up and concave down.

9. Determine where the following functions are increasing, decreasing, concave up, and concave down. Find all critical numbers, local extrema, and inflection points. Find any asymptotes. Sketch the graph of the function.
(a) \( f(x) = \frac{x^2}{x^2 + 1} \)  
(b) \( f(x) = x^3 - 2x^2 + x + 1 \)
(c) \( f(x) = \frac{x^2 - 1}{x^3} \)

10. (a) Sketch a graph of a continuous function that satisfies all of the following conditions.
Domain: \([0, 6]\)
- \( f(0) = 3; \quad f(3) = 0; \quad f(6) = 4 \)
- \( f'(x) < 0 \) on \((0, 3); \quad f'(x) > 0 \) on \((3, 6)\)
- \( f''(x) > 0 \) on \((0, 5); \quad f''(x) < 0 \) on \((5, 6)\)

(b) Sketch a graph of a continuous function that satisfies all of the following conditions.
Domain: \((-3, \infty)\)
- \( \lim_{x\to-3^+} f(x) = \infty \)
- \( \lim_{x\to\infty} f(x) = 1 \)
- \( f'(1) = 0; \quad f'(x) < 0 \) on \((-3, 1); \quad f'(x) > 0 \) on \((1, \infty)\)
- \( f''(2) = 0; \quad f''(x) > 0 \) on \((-3, 2); \quad f''(x) < 0 \) on \((2, \infty)\)
11. Below is the graph of the derivative $f'(x)$ of a function $f$. Determine the intervals on which the graph of $f$ is increasing, decreasing, concave up, and concave down. Find the critical numbers. Find the $x$-coordinate of the inflection points.

![Graph of $y = f'(x)$](image)

12. Solve the following optimization problems.

(a) A farmer wishes to fence off two identical adjoining rectangular pens, each with 900 square feet of area, as shown in the diagram below. What are the $x$- and $y$-values so that the least amount of fence is required?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
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(b) A handbill is to contain 50 square inches of printed matter, with 4-inch margins at the top and bottom and 2-inch margins on each side. What dimensions for the handbill would use the least paper?

(c) A cylindrical biscuit can is to be made with a cardboard side and metal ends. Assume that it costs twice as much per unit area to construct the ends as it does the side. Find the dimensions that minimize the cost of the can if the volume is to be $256\pi$ cubic centimeters.

(d) A small island is 2 miles from the nearest point $P$ on the straight shoreline of a large lake. If a woman on the island has a motorboat that goes 20 mph and can walk 4 miles per hour, where should the boat be landed in order to arrive at a town 10 miles down the shore from $P$ in the least time?

(e) A rectangular box is to be made by cutting out equal squares from each corner of a piece of cardboard 10 inches by 16 inches and then folding up the sides. What must be the length of the side of the square cut out if the volume is to be maximized? What is the maximum volume?
13. Find the indicated limits analytically (using algebra).

(a) \( \lim_{x \to \infty} \frac{2x^2}{3x^2 + 5} \)  
(b) \( \lim_{x \to \infty} \frac{2x}{3x^2 + 5} \)

(c) \( \lim_{x \to \infty} \frac{5 \cos x}{x} \)  
(d) \( \lim_{x \to \infty} \frac{3x}{\sqrt{4x^2 + 1}} \)

(e) \( \lim_{x \to \infty} \frac{5x^2}{x + 3} \)  
(f) \( \lim_{x \to \infty} \frac{3x^4 + 3x^2 + x}{5x^2 - 2x + 10} \)

ANSWERS

1. (a) \( dy = (21x^2 - 6x)dx \)  
(b) \( dy = \frac{dx}{(1-x)^2} \)

(c) \( dy = \sec x \tan x \, dx \)

2. (a) \( \Delta y \approx 2.25; \, dy = 2 \)  
(b) \( \Delta y \approx 0.073; \, dy = \frac{3}{40} = 0.075 \)

3. \( L(x) = 5 + \frac{4}{5}(x - 4); \, \sqrt{25.81} \approx 5.08 \)

4. (a) \( 8000\pi \text{ cm}^3 \approx 25,133 \text{ cm}^3 \)  
(b) \( 102.6 \text{ in}^3 \)

(c) \( 5\pi \text{ in}^2 \approx 15.7 \text{ in}^2 \)

5. (a) min: \( f(1) = -1; \, \text{max: } f(3) = 19 \)  
(b) min: \( f(-1) = -\frac{1}{3}; \, \text{max: } f(\sqrt{2}) = \frac{\sqrt{2}}{4} \)

(c) min: \( f(0) = 0; \, \text{max: } f(32) = 4 \)

6. (a) \( c = 1 \)  
(b) \( c = \frac{25}{4} \)  
(d) \( c = \frac{\pi}{2} \)

(e) \( f \) is not continuous on \([-1, 1] \)

7. (a) local min: \( f(0) = 0; \, \text{local max: } f\left(-\frac{1}{27}\right) = \frac{1}{27} \)  
(b) local min: \( f(-1) = -6; \, \text{local max: } f(2) = 21 \)

(c) local min: \( f(1) = 2; \, \text{local max: } f(-1) = -2 \)

(d) local min: \( F\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{3}; \, \text{local max: } f\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{3} \)

8. (a) increasing: \((1, 3); \, \text{decreasing: } (-\infty, 1) \text{ and } (3, \infty) \]

(b) concave up: \((2, \infty); \, \text{concave down: } (-\infty, 2); \, \text{inflection point: } (2, -48) \)
9. (a) increasing: \((0, \infty)\); decreasing: \((-\infty, 0)\); concave up: \((-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\); concave down: \((-\infty, -\frac{1}{\sqrt{3}})\) and \(\left(\frac{1}{\sqrt{3}}, \infty\right)\); critical number: \(x = 0\); local min: \(f(0) = 0\); inflections points: \(\left(\pm \frac{1}{\sqrt{3}}, \frac{1}{4}\right)\); horizontal asymptote: \(y = 1\)

(b) increasing: \((-\infty, \frac{1}{3})\) and \((1, \infty)\); decreasing: \(\left(\frac{1}{3}, 1\right)\); concave up: \(\left(\frac{2}{3}, \infty\right)\); concave down: \((-\infty, \frac{2}{3})\); critical numbers: \(x = \frac{1}{3}, 1\); local max: \(f\left(\frac{1}{3}\right) = \frac{31}{27}\); local min: \(f(1) = 1\); inflection point: \(\left(\frac{2}{3}, \frac{29}{27}\right)\)

(c) increasing: \((-\sqrt{3}, 0)\) and \((0, \sqrt{3})\); decreasing: \((-\infty, \sqrt{3})\) and \((\sqrt{3}, \infty)\); concave up: \((-\sqrt{6}, 0)\) and \((\sqrt{6}, \infty)\); concave down: \((-\infty, -\sqrt{6})\) and \((0, \sqrt{6})\); critical numbers: \(x = \sqrt{3}, 0, \sqrt{3}\); local min: \(f(-\sqrt{3}) = -\frac{2}{3\sqrt{3}}\); local max: \(f(\sqrt{3}) = \frac{2}{3\sqrt{3}}\); inflection points: \(\left(\pm \sqrt{6}, \pm \frac{5}{6\sqrt{6}}\right)\); vertical asymptote: \(x = 0\); horizontal asymptote: \(y = 0\)

10. various correct answers

11. increasing: \((-\infty, -2)\) and \((0, 2)\); decreasing: \((-2, 0)\) and \((2, \infty)\); concave up: \((-1, 1)\); concave down: \((-\infty, -1)\) and \((1, \infty)\); critical numbers: \(x = -2, 0, 2\); inflection points at \(x = -1, 1\).

12. (a) \(x = 15\sqrt{3}\) ft; \(y = 20\sqrt{3}\) ft
(b) 9 inches by 18 inches
(c) radius = 4 cm; height = 16 cm
(d) The woman should drive the boat directly to town.
(e) square cut-out: 2 inches; volume: 144 \(\text{in}^3\)

13. (a) \(\frac{2}{3}\)
(b) 0
(c) 0
(d) \(\frac{3}{2}\)
(e) \(-\infty\)
(f) \(\infty\)