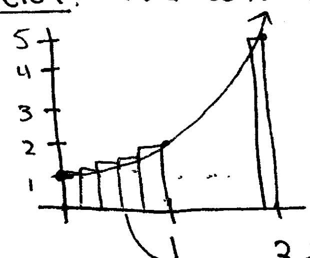
Examples for Section 4.1

Example: Find a formula for L_n and R_n, area of the left and right rectangles, respectively, to approximate the area between $f(x)=x^2+1$ and the x-axis from $x=0$ to $x=2$. Find the area by evaluating $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} R_n$.

Solution: We will work with R_n first



each rectangle has width $\Delta x = \frac{2}{n}$

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

We form the subintervals by using

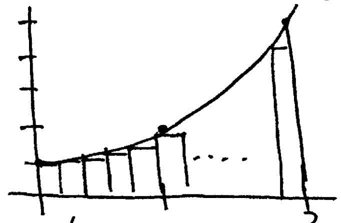
$$x_0 = 0$$
$$x_1 = 0 + \frac{2}{n}$$
$$x_2 = x_1 + \frac{2}{n} = \frac{2}{n} + \frac{2}{n} = \frac{4}{n}$$
$$x_3 = x_2 + \frac{2}{n} = \frac{4}{n} + \frac{2}{n} = \frac{6}{n}$$
$$\vdots$$
$$x_i = \frac{2i}{n}$$
$$\vdots$$
$$x_n = 2$$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n}$$
$$= \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + 1\right]\frac{2}{n}$$
$$= \sum_{i=1}^{n} \left[\frac{4i^2}{n^2}+1\right]\frac{2}{n} = \sum_{i=1}^{n}\left[\frac{8i^2}{n^3}+\frac{2}{n}\right] = \sum_{i=1}^{n}\frac{8i^2}{n^3} + \sum_{i=1}^{n}\frac{2}{n}$$
$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n}\frac{2}{n}$$
$$= \frac{\cancel{8}^{\,4}}{n^3}\cdot\frac{n(n+1)(2n+1)}{\cancel{6}\,3} + \cancel{n}\cdot\frac{2}{\cancel{n}}$$
$$= \frac{4}{3n^3}\cdot(2n^3+3n^2+n) + 2$$
$$= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} + 2$$

Area under curve $= A = \lim_{n\to\infty} R_n$

$$= \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}+2\right)$$
$$= \frac{8}{3}+0+0+2 = \frac{14}{3}$$

Now, let's work with L_n



$\hookrightarrow$ each rectangle has width $\Delta x = \frac{2}{n}$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$x_0 = 0$

$x_1 = x_0 + \frac{2}{n} = 0 + \frac{2}{n} = \frac{2}{n}$

$x_2 = x_1 + \frac{2}{n} = \frac{2}{n} + \frac{2}{n} = \frac{4}{n}$

$\vdots$

$x_i = \frac{2i}{n}$

$\vdots$

$x_n = 2$

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left(\frac{2(i-1)}{n}\right)\frac{2}{n}$$

$$= \sum_{i=1}^{n} \left[\left(\frac{2(i-1)}{n}\right)^2 + 1\right]\frac{2}{n}$$

$$= \sum_{i=1}^{n} \left[\frac{4(i-1)^2}{n^2} + 1\right]\frac{2}{n}$$

$$= \sum_{i=1}^{n} \left[\frac{8(i-1)^2}{n^3} + \frac{2}{n}\right]$$

$$= \sum_{i=1}^{n} \frac{8(i-1)^2}{n^3} + \sum_{i=1}^{n} \frac{2}{n}$$

$$= \frac{8}{n^3}\sum_{i=1}^{n}(i-1)^2 + \sum_{i=1}^{n}\frac{2}{n}$$

$$= \frac{8}{n^3}\sum_{i=1}^{n}(i^2-2i+1) + \sum_{i=1}^{n}\frac{2}{n} = \frac{8}{n^3}\left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right] + \sum_{i=1}^{n}\frac{2}{n}$$

$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \cancel{2}\cdot\frac{n(n+1)}{\cancel{2}} + n\right] + n\cdot\frac{2}{n}$$

$$= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6} - (n^2+n) + n\right] + 2$$

$$= \frac{8}{n^3}\left[\frac{\cancel{2}n^3}{\cancel{6}3} + \frac{\cancel{3}n^2}{2\cancel{6}} + \frac{n}{6} - n^2 - \cancel{n} + \cancel{n}\right] + 2$$

$$= \frac{8n^3}{3n^3} + \frac{4n^2}{n^3} + \frac{8n}{6n^3} - \frac{8n^2}{n^3} + 2$$

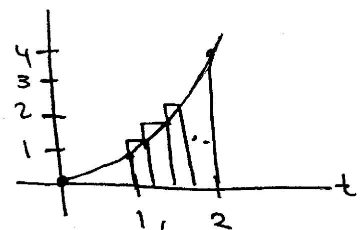
$$= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} - \frac{8}{n} + 2$$

$$\text{Area} = A = \lim_{n\to\infty} L_n = \lim_{n\to\infty}\left[\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} - \frac{8}{n} + 2\right]$$

$$= \frac{8}{3} + 0 + 0 - 0 + 2 = \frac{14}{3}$$

Example: Suppose an object travels with velocity $v(t) = t^2$ measured in feet per second. How far did the object travel between $t = 1$ second and $t = 2$ seconds?

Solution



↳ each rectangle has width $\Delta t = 1/n$

$$\Delta t = \frac{2-1}{n} = \frac{1}{n}$$

$t_0 = 1$

$t_1 = t_0 + \frac{1}{n} = 1 + \frac{1}{n}$

$t_2 = t_1 + \frac{1}{n} = 1 + \frac{1}{n} + \frac{1}{n} = 1 + \frac{2}{n}$

$t_3 = t_2 + \frac{1}{n} = 1 + \frac{2}{n} + \frac{1}{n} = 1 + \frac{3}{n}$

$\vdots$

$t_i = 1 + \frac{i}{n}$

$\vdots$

$t_n = 2$

NOTE: $\Delta s_i = v(t_i)\Delta t$

$$
\begin{aligned}
R_n &= \sum_{i=1}^{n} v(t_i)\Delta t \\
&= \sum_{i=1}^{n} v\left(1+\frac{i}{n}\right)\cdot\frac{1}{n} \\
&= \sum_{i=1}^{n} \left[\left(1+\frac{i}{n}\right)^2\right]\cdot\frac{1}{n} \\
&= \sum_{i=1}^{n} \left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)\frac{1}{n} \\
&= \sum_{i=1}^{n} \left(\frac{1}{n}+\frac{2i}{n^2}+\frac{i^2}{n^3}\right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{n}\right) + \sum_{i=1}^{n}\frac{2i}{n^2} + \sum_{i=1}^{n}\frac{i^2}{n^3} \\
&= \sum_{i=1}^{n}\frac{1}{n} + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= n\cdot\frac{1}{n} + \frac{\cancel{2}}{n^2}\cdot\frac{n(n+1)}{\cancel{2}} + \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\
&= 1 + \frac{1}{n^2}(n^2+n) + \frac{1}{6n^3}(2n^3+3n^2+n) = \\
&= 1 + \frac{n^2}{n^2} + \frac{n}{n^2} + \frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3} \\
&= 1 + 1 + \frac{1}{n} + \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \\
&= \frac{7}{3} + \frac{3}{2n} + \frac{1}{6n^2}
\end{aligned}
$$

$$
\begin{aligned}
s &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty}\left(\frac{7}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) \\
&= \frac{7}{3} + 0 + 0 = \frac{7}{3}
\end{aligned}
$$

Thus, the object traveled $\frac{7}{3}$ feet between $t = 1$ second and $t = 2$ seconds.