Examples for Section 3.5

MATH 1501

① $y = \dfrac{x^2}{x^2-9}$

STEP 1:

(a) Domain: all real numbers except where denominator is zero.

$x^2-9=0 \Rightarrow (x+3)(x-3)=0 \Rightarrow x+3=0,\ x-3=0 \Rightarrow x=-3,\ x=3$

So, domain is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$

(b) y-intercept: $y = \dfrac{0^2}{0^2-9} = \dfrac{0}{-9} = 0$ $(0,0)$ is y-intercept

x-intercept: $0 = \dfrac{x^2}{x^2-9} \Rightarrow (x^2-9)0 = x^2 \Rightarrow 0 = x^2 \Rightarrow 0 = x$

So, $(0,0)$ is also the x-intercept

(c) If $f(x)$ is even, it is symmetric with respect to the y-axis

If $f(x)$ is odd, it is symmetric with respect to the origin

Even means $f(-x)=f(x)$; odd means $f(-x)=-f(x)$

$f(-x) = \dfrac{(-x)^2}{(-x)^2-9} = \dfrac{x^2}{x^2-9}$ So, even and symm wrt y-axis

STEP 2

(a) Asymptotes

$$\left.\begin{aligned}
&\lim_{x\to\infty} \frac{x^2}{x^2-9}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1}{1-\frac{9}{x^2}} = \frac{1}{1-0} = 1\\
&\lim_{x\to-\infty} \frac{x^2}{x^2-9}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty}\frac{1}{1-\frac{9}{x^2}} = \frac{1}{1-0} = 1
\end{aligned}\right\}\text{ So, } y=1 \text{ is h.a}$$

$$\left.\begin{aligned}
&\lim_{x\to-3^-} \frac{x^2}{x^2-9} = \frac{9}{0^+} = \infty\\
&\lim_{x\to-3^+} \frac{x^2}{x^2-9} = \frac{9}{0^-} = -\infty
\end{aligned}\right\}\text{ So, } x=-3 \text{ is a v.a}$$

$$\left.\begin{aligned}
&\lim_{x\to3^-} \frac{x^2}{x^2-9} = \frac{9}{0^-} = -\infty\\
&\lim_{x\to3^+} \frac{x^2}{x^2-9} = \frac{9}{0^+} = \infty
\end{aligned}\right\}\text{ So, } x=3 \text{ is a v.a}$$

(b) Increasing/Decreasing

$$f'(x) = \frac{(x^2-9)(2x) - x^2(2x)}{(x^2-9)^2} = \frac{2x^3 - 18x - 2x^3}{(x^2-9)^2} = -\frac{18x}{(x^2-9)^2}$$

$$\frac{-18x}{(x^2-9)^2} = 0 \Rightarrow -18x = 0(x^2-9)^2 \Rightarrow -18x = 0 \Rightarrow x = 0$$

$f'(x)$ is undefined where $x^2-9=0$ or $x = \pm 3$

So, critical numbers are $-3, 0, 3$

inc	inc	dec	dec
-4	-1	1	4
$f'(-4) = \frac{72}{49}$	$f'(-1) = \frac{18}{64}$	$f'(1) = -\frac{18}{64}$	$f'(4) = -\frac{72}{49}$

(dividing points on the number line: -3, 0, 3)

inc: $(-\infty, -3)$ and $(-3, 0)$

dec: $(0, 3)$ and $(3, \infty)$

(c) local exetrema: $(0,0)$ is a local max

(d) concavity / inflection points

$$f''(x) = \frac{(x^2-9)^2(-18) - (-18x)\,2(x^2-9)(2x)}{[(x^2-9)^2]^2} = \frac{-18(x^2-9)[(x^2-9) - 4x^2]}{(x^2-9)^4}$$

$$= \frac{-18(-3x^2-9)}{(x^2-9)^3} = \frac{-18(-3)(x^2+3)}{(x^2-9)^3} = \frac{54(x^2+3)}{(x^2-9)^3}$$

$$\frac{54(x^2+3)}{(x^2-9)^3} = 0 \Rightarrow 54(x^2+3) = 0 \Rightarrow x^2+3 = 0 \Rightarrow x^2 = -3$$

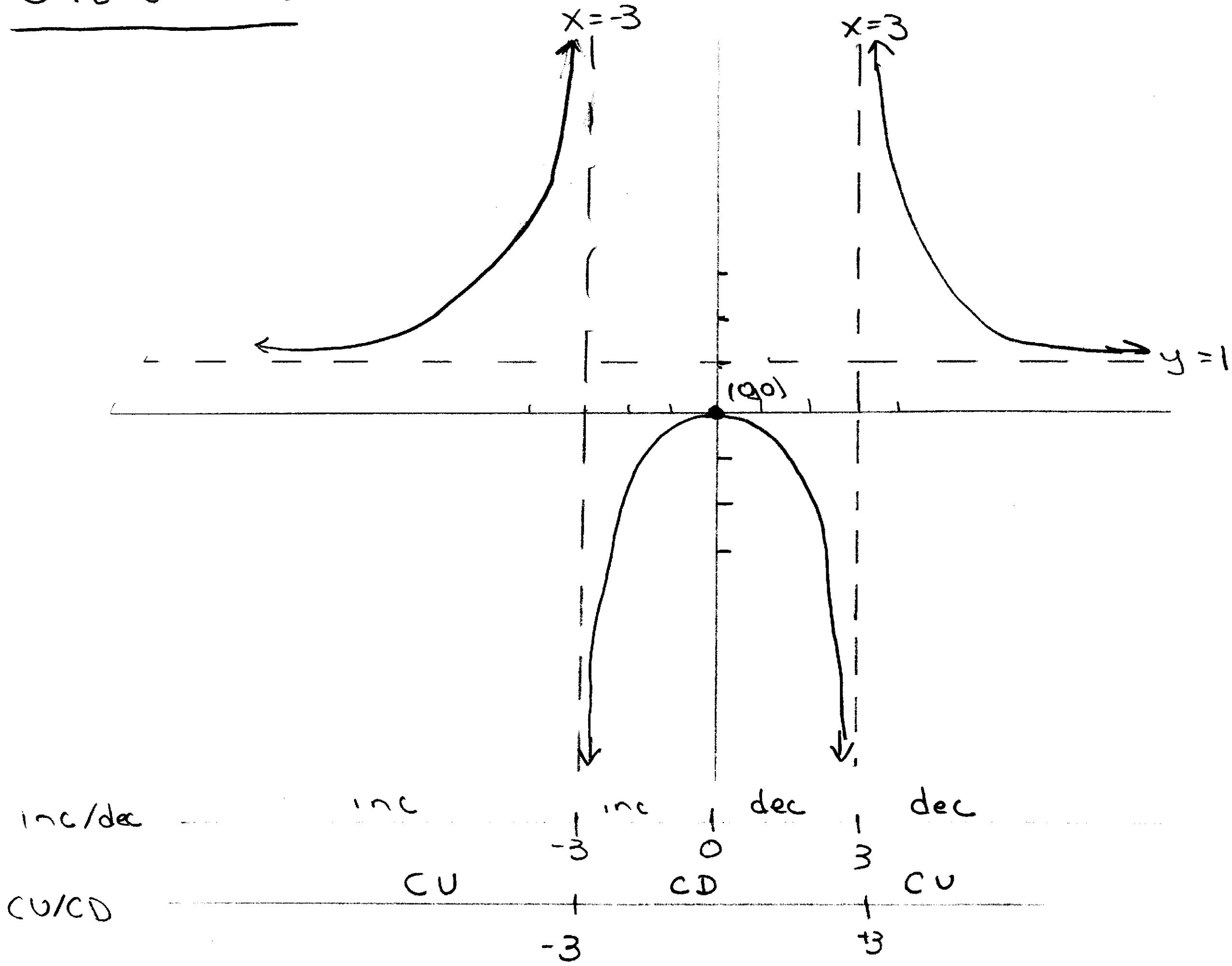
So, no solution

But denominator is zero when $x = -3, 3$

CU	CD	
-4	0	4
$f''(-4) = \frac{1026}{343}$	$f''(0) = -\frac{2}{9}$	$f''(4) = \frac{1026}{343}$

(dividing points on the number line: -3, 3)

STEPS 3 + 4



(2) $y = \dfrac{x-2}{x^4}$

STEP 1:

(a) Domain: $x^4 = 0 \Rightarrow x = 0$

$(-\infty, 0) \cup (0, \infty)$

(b) y-intercept: none (division by zero: $y = \dfrac{0-2}{0}$ undefined)

x-intercept: $0 = \dfrac{x-2}{x^4} \Rightarrow 0 = x-2 \Rightarrow 2 = 0$

x-intercept: $(2, 0)$

(c) $f(-x) = \dfrac{-x-2}{(-x)^4} = \dfrac{-x-2}{x^4}$ no symmetry

STEP 2

(a) asymptotes

$$\left.\begin{array}{l} \lim\limits_{x\to\infty} \dfrac{x-2}{x^4} \cdot \dfrac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x^3} - \frac{2}{x^4}}{1} = \dfrac{0-0}{1} = 0 \\ \lim\limits_{x\to\infty} \dfrac{x-2}{x^4} \cdot \dfrac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x^3} - \frac{2}{x^4}}{1} = \dfrac{0-0}{1} = 0 \end{array}\right\} \text{h.a: } y = 0$$

$$\left.\begin{array}{l} \lim\limits_{x\to 0^-} \dfrac{x-2}{x^4} = \dfrac{-2}{0^+} = -\infty \\ \lim\limits_{x\to 0^+} \dfrac{x-2}{x^4} = \dfrac{-2}{0^+} = -\infty \end{array}\right\} \text{v.a: } x = 0$$

(b) increasing/ decreasing

$$f'(x) = \frac{x^4 \cdot 1 - (x-2)(4x^3)}{(x^4)^2} = \frac{x^3[x - (x-2)4]}{x^8} = \frac{x - 4x + 8}{x^5} = \frac{-3x+8}{x^5}$$

$$\frac{-3x+8}{x^5} = 0 \Rightarrow -3x+8 = 0 \Rightarrow x = \frac{8}{3}$$

critical numbers are: $\frac{8}{3}$ + 0 (where f' is undefined)

dec | inc | dec

$-1 \quad 0 \quad 1 \quad \frac{8}{3} \quad 3$

$f'(-1) = -11$ | $f'(1) = 5$ | $f'(3) = -\frac{1}{243}$

increasing: $(0, \frac{8}{3})$

decreasing: $(-\infty, 0)$ and $(\frac{8}{3}, \infty)$

(c) local extrema:

$$f\left(\tfrac{8}{3}\right) = \frac{\frac{8}{3} - 2}{\left(\frac{8}{3}\right)^4} = \frac{27}{2048} \quad \text{is a local max}$$

Since there is a v.a at $x=0$, $f(0)$ is undefined + therefore cannot be a local min or local max.

(d) concavity + inflection points

$$f''(x) = \frac{x^5(-3) - (-3x+8)5x^4}{x^{10}} = \frac{x^4\left[-3x - (-3x+8)5\right]}{x^{10}}$$

$$= \frac{-3x + 15x - 40}{x^6} = \frac{12x - 40}{x^6} = \frac{4(3x-10)}{x^6}$$

$$\frac{4(3x-10)}{x^6} = 0 \Rightarrow 3x - 10 = 0 \Rightarrow 3x = 10 \Rightarrow x = \frac{10}{3}$$

Also, include $x=0$ since f'' is undefined at $x=0$

CD | CD | CD

-1 | 0 | 1 | $\frac{10}{3}$ | 4

$f''(-1) = -52$ | $f''(1) = -28$ | $f''(4) = \frac{1}{512}$

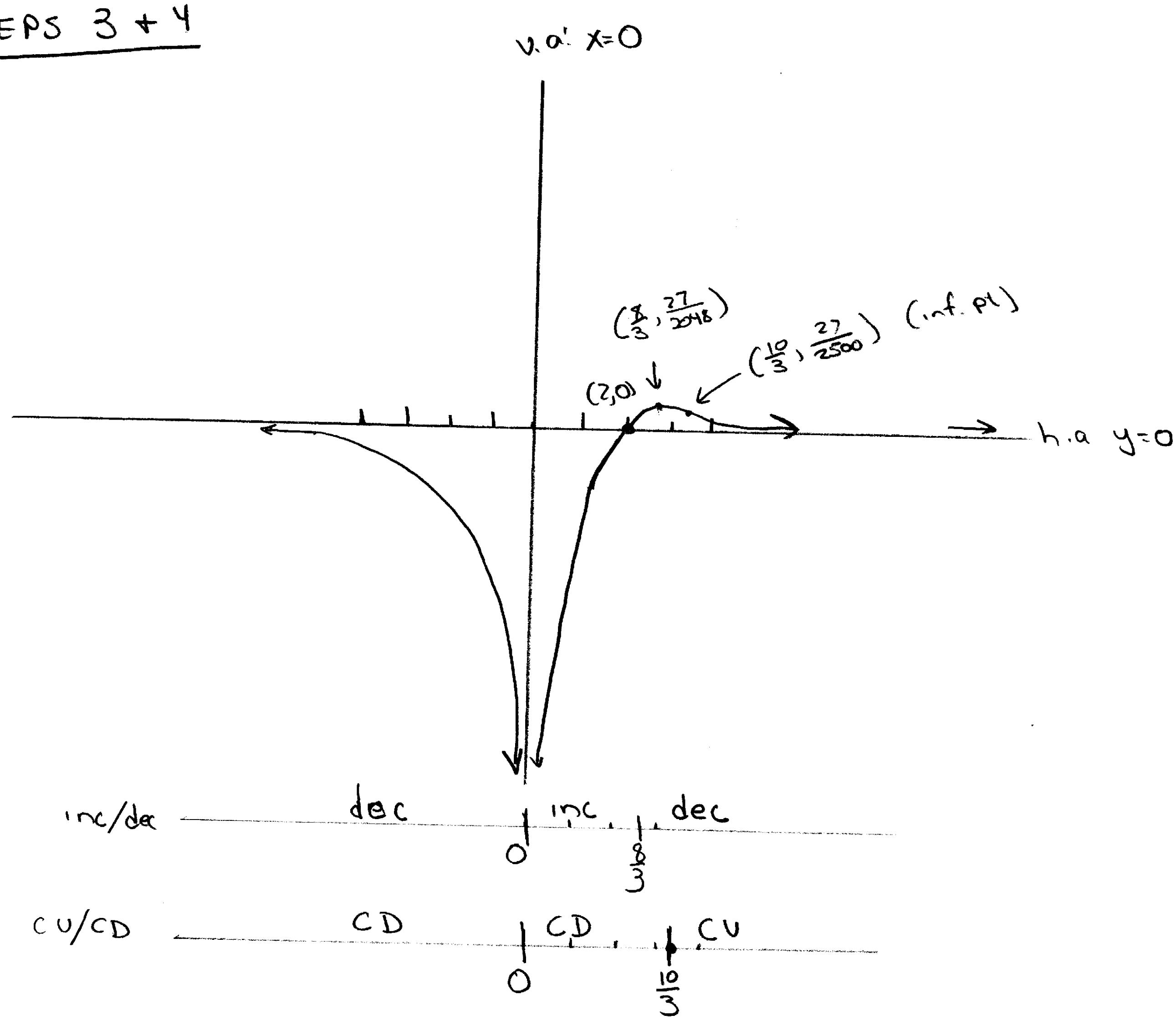
CD: $(-\infty, 0)$ and $(0, \frac{10}{3})$

CU: $(\frac{10}{3}, \infty)$

$$f\left(\tfrac{10}{3}\right) = \frac{\frac{10}{3} - 2}{\left(\frac{10}{3}\right)^4} = \frac{27}{2500}$$

So, $\left(\frac{10}{3}, \frac{27}{2500}\right)$ is an inflection point

STEPS 3 + 4



NOTE: If you graph this function on your calculator, you will probably not find the local max. The calculus analysis yields information that is sometimes not obvious from technology.

(3) $y = \frac{x^2+12}{x-2}$

STEP 1:

(a) Domain $x-2=0 \Rightarrow x=2$

D: $(-\infty, 2) \cup (2, \infty)$

(b) y-intercept: $y = \frac{0^2+12}{0-2} = \frac{12}{-2} = -6$ $(0,-6)$ y-int

x-intercept: $0 = \frac{x^2+12}{x-2} \Rightarrow 0 = x^2+12 \Rightarrow -12 = x^2$

No solution; so no x-int

(c) $f(-x) = \frac{(-x)^2+12}{-x-2} = \frac{x^2+12}{-x-2}$ no symmetry

STEP 2:

(a) $\lim_{x\to\infty} \frac{x^2+12}{x-2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x+\frac{12}{x}}{1-\frac{2}{x}} = \frac{\infty+0}{1-0} = \infty$

$\lim_{x\to-\infty} \frac{x^2+12}{x-2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{x+\frac{12}{x}}{1-\frac{2}{x}} = \frac{-\infty+0}{1-0} = -\infty$

$\Big\}$ no. h.a

$\lim_{x\to2^-} \frac{x^2+12}{x-2} = \frac{16}{0^-} = -\infty$

$\lim_{x\to2^+} \frac{x^2+12}{x-2} = \frac{16}{0^+} = \infty$

$\Big\}$ V.a: $x=2$

Since degree of numerator is exactly on large than degree of the denominator, there is a slant asymptote. We find it by long division

$$
\begin{array}{r l}
 & x + 2 \quad r: 6 \\
x-2 \, \big) & x^2 + 0x + 12 \\
 & -(x^2 - 2x) \\
\hline
 & \quad 2x + 12 \\
 & -(2x - 4) \\
\hline
 & \quad\quad 16
\end{array}
$$

S.a is $y = x+2$

(b) $f'(x) = \dfrac{(x-2)\cdot 2x - (x^2+12)\cdot 1}{(x-2)^2} = \dfrac{2x^2 - 4x - x^2 - 12}{(x-2)^2}$

$= \dfrac{x^2 - 4x - 12}{(x-2)^2} = \dfrac{(x-6)(x+2)}{(x-2)^2}$

Critical numbers: 6, -2, 2

inc | dec | dec | inc

$-3 \quad -2 \quad 0 \quad 2 \quad 3 \quad 6 \quad 5$

$f'(-3) = \frac{3}{25}$ | $f'(0) = 3$ | $f'(3) = -9$ | $f'(5) = \frac{1}{3}$

inc: $(-\infty, -2)$ and $(6, \infty)$

dec: $(-2, 2)$ and $(2, 6)$

(c) local extrema

$f(-2) = \dfrac{(-2)^2 + 12}{-2-2} = -4$ is a local max

$f(6) = \dfrac{6^2 + 12}{6-2} = 12$ is a local min

(d) concavity / inflection points

$f''(x) = \dfrac{(x-2)^2(2x-4) - (x^2-4x-12)\cdot 2(x-2)\cdot 1}{(x-2)^4}$

$= \dfrac{(x-2)\left[(x-2)(2x-4) - (x^2-4x-12)\cdot 2\right]}{(x-2)^4}$

$= \dfrac{2x^2 - 8x + 8 \; - 2x^2 + 8x + 24}{(x-2)^3}$

$= \dfrac{32}{(x-2)^3}$

only use $x = 2$

CD | CU

$1 \quad 2 \quad 3$

$f''(1) = -32$ | $f''(3) = 32$

CD: $(-\infty, 2)$
CU: $(2, \infty)$

STEPS 3 and 4