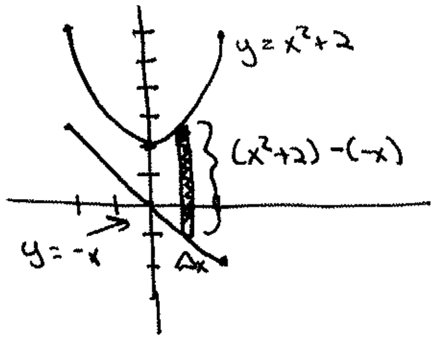
Examples for Section 5.1

① Find the area between $y=x^2+2$ and $y=-x$ from $x=-2$ to $x=2$



$$A = \int_{-2}^{2}\left[(x^2+2)-(-x)\right]dx$$

$$= \int_{-2}^{2}\left[x^2+x+2\right]dx$$

$$= \left[\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+2x\right]_{-2}^{2}$$

$$= \left[\tfrac{1}{3}(2)^3+\tfrac{1}{2}(2)^2+2\cdot 2\right]-\left[\tfrac{1}{3}(-2)^3+\tfrac{1}{2}(-2)^2+2(-2)\right]$$

$$= \left(\tfrac{8}{3}+2+4\right)-\left(-\tfrac{8}{3}+2-4\right)$$

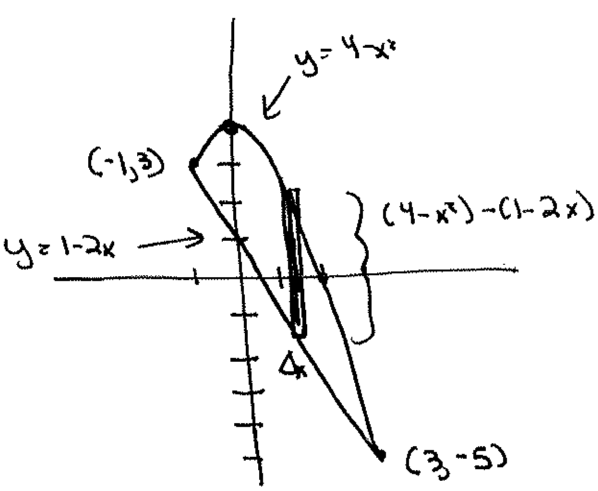
$$= \tfrac{8}{3}\cancel{+2}+4+\tfrac{8}{3}\cancel{-2}+4 = \tfrac{16}{3}+8 = \tfrac{40}{3}$$

② Find the area enclosed by $y=4-x^2$ and $y=1-2x$

NOTE: we need the intersection points.

$$4-x^2 = 1-2x$$
$$0 = 1-2x-4+x^2$$
$$0 = x^2-2x-3$$
$$0 = (x-3)(x+1)$$
$$x=3,\ x=-1$$

Graphs intersect at $x=3$ $x=-1$



$$A = \int_{-1}^{3}\left[(4-x^2)-(1-2x)\right]dx$$

$$= \int_{-1}^{3}(4-x^2-1+2x)\,dx$$

$$= \int_{-1}^{3}(3+2x-x^2)\,dx$$

$$= \left[3x+x^2-\tfrac{1}{3}x^3\right]_{-1}^{3}$$

$$= \left[3\cdot 3+3^2-\tfrac{1}{3}(3)^3\right]-\left[3(-1)+(-1)^2-\tfrac{1}{3}(-1)^3\right]$$

$$= (9\cancel{+9}\cancel{-9})-\left(-3+1+\tfrac{1}{3}\right)$$

$$= 9+3-1-\tfrac{1}{3}$$

$$= 11-\tfrac{1}{3}$$

$$= \tfrac{32}{3}$$

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$y = \cos x, \quad y = \sin 2x \qquad x = 0, \; x = \frac{\pi}{2}$

Find intersection points

$\sin 2x = \cos x$

$\sin 2x - \cos x = 0$

$2\sin x \cos x - \cos x = 0$

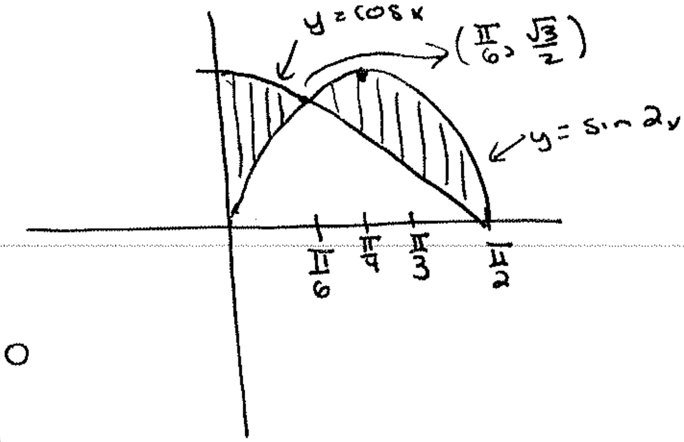
$\cos x (2\sin x - 1) = 0$

$\cos x = 0 \qquad 2\sin x - 1 = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \qquad 2\sin x = 1$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$



$$A = \int_0^{\pi/2} |\cos x - \sin 2x| \, dx$$

$$= \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx$$

$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$

$$= \left[\left(\sin\tfrac{\pi}{6} + \tfrac{1}{2}\cos 2\cdot\tfrac{\pi}{6}\right) - \left(\sin 0 + \tfrac{1}{2}\cos 2\cdot 0\right)\right]$$
$$+ \left[\left(-\tfrac{1}{2}\cos 2\cdot\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\right) - \left(-\tfrac{1}{2}\cos 2\cdot\tfrac{\pi}{6} - \sin\tfrac{\pi}{6}\right)\right]$$

$$= \left[\left(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2}\right) - \left(0 + \tfrac{1}{2}\cdot 1\right)\right] + \left[\left(-\tfrac{1}{2}(-1) - 1\right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right)\right]$$

$$= \left(\tfrac{3}{4} - \tfrac{1}{2}\right) + \left[\left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{4} - \tfrac{1}{2}\right)\right]$$

$$= \tfrac{1}{4} + \left(-\tfrac{1}{2} + \tfrac{3}{4}\right)$$

$$= \tfrac{1}{4} + \tfrac{1}{4}$$

$$= \tfrac{2}{4}$$

$$= \tfrac{1}{2}$$

Note: $\int \sin 2x \, dx$

$u = 2x$

$du = 2dx$

$\frac{1}{2} du = dx$

$= \frac{1}{2}\int \sin u \, du$

$= \frac{1}{2}(-\cos u) + C$

$= -\frac{1}{2}\cos 2x + C$

(4) Find the area enclosed by the line $y = x-1$ and the parabola $x = 3-y^2$.

Find points of intersection:

$$y = x-1 \Rightarrow y+1 = x$$

$$y+1 = 3-y^2$$

$$y^2+y+1-3 = 0$$

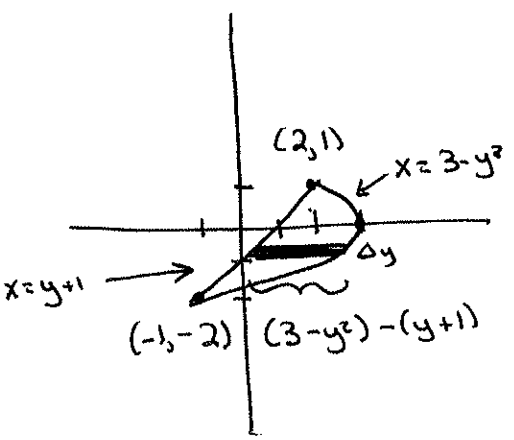
$$y^2+y-2 = 0$$

$$(y+2)(y-1) = 0$$

$$y = -2 \quad y = 1$$

$$y=-2 \Rightarrow x = -2+1 = -1$$

$$y = 1 \Rightarrow x = 1+1 = 2$$



$$A = \int_{-2}^{1} [(3-y^2)-(y+1)]\,dy$$

$$= \int_{-2}^{1} [3-y^2-y-1]\,dy$$

$$= \int_{-2}^{1} [2-y-y^2]\,dy$$

$$= \left[2y - \tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_{-2}^{1}$$

$$= \left[2\cdot 1 - \tfrac{1}{2}\cdot 1^2 - \tfrac{1}{3}\cdot 1\right] - \left[2(-2) - \tfrac{1}{2}(-2)^2 - \tfrac{1}{3}(-2)^3\right]$$

$$= \left[2 - \tfrac{1}{2} - \tfrac{1}{3}\right] - \left[-4 - 2 + \tfrac{8}{3}\right]$$

$$= 2 - \tfrac{1}{2} - \tfrac{1}{3} + 6 - \tfrac{8}{3}$$

$$= 8 - \tfrac{1}{2} - \tfrac{9}{3}$$

$$= 8 - \tfrac{1}{2} - 3$$

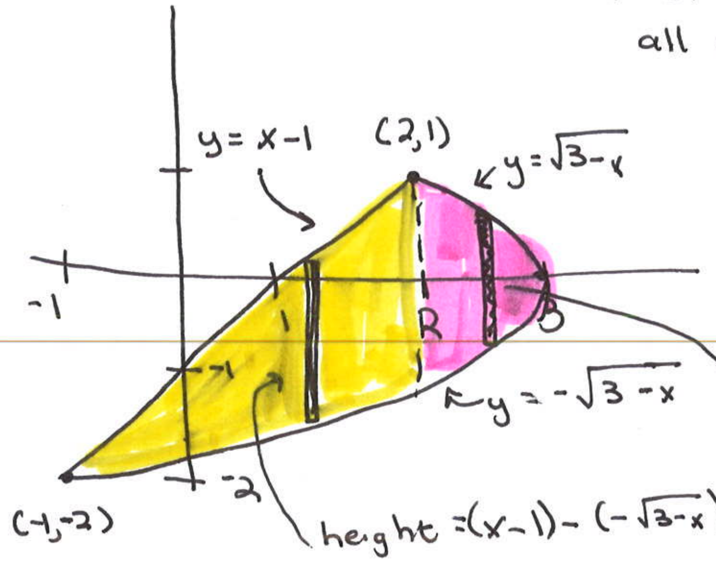
$$= 5 - \tfrac{1}{2}$$

$$= \tfrac{9}{2}$$

On the next page is this same problem set up and solved using vertical slices. However, as you will see, horizontal slices are easier in this case

Alternative Set Up and Solution to Example 4

Note: Since the slices are vertical, we need all equations in terms of "x".



$x = 3 - y^2$ $\qquad$ $y = x - 1$

$y^2 + x = 3$

$y^2 = 3 - x$

$y = \pm\sqrt{3-x}$

height $= \sqrt{3-x} - (-\sqrt{3-x})$

<u>Note</u>: Since the "top" function is different, we need two integrals.

$$A = \int_{-1}^{2} \left[(x-1) - (-\sqrt{3-x})\right] dx + \int_{2}^{3} \left[\sqrt{3-x} - (-\sqrt{3-x})\right] dx$$

$$= \int_{-1}^{2} (x - 1 + \sqrt{3-x})\, dx + \int_{2}^{3} (\sqrt{3-x} + \sqrt{3-x})\, dx$$

$$= \int_{-1}^{2} (x-1)\, dx + \int_{-1}^{2} \sqrt{3-x}\, dx + \int_{2}^{3} 2\sqrt{3-x}\, dx$$

$u = 3 - x$

$du = -1\, dx$

$-du = dx$

$$= \int_{-1}^{2} (x-1)\, dx - \int_{4}^{1} \sqrt{u}\, du - 2\int_{1}^{0} \sqrt{u}\, du$$

$$= \int_{-1}^{2} (x-1)\, dx + \int_{1}^{4} u^{1/2}\, du + 2\int_{0}^{1} u^{1/2}\, du$$

$$= \left[\frac{1}{2}x^2 - x\right]_{-1}^{2} + \left[\frac{2}{3}u^{3/2}\right]_{1}^{4} + 2\left[\frac{2}{3}u^{3/2}\right]_{0}^{1}$$

$$= \left[\left(\frac{1}{2}\cdot 2^2 - 2\right) - \left(\frac{1}{2}(-1)^2 - (-1)\right)\right] + \left[\frac{2}{3}\cdot 4^{3/2} - \frac{2}{3}\cdot 1^{3/2}\right] + 2\left[\frac{2}{3}\cdot 1^{3/2} - \frac{2}{3}\cdot 0\right]$$

$$= \left[(2-2) - \left(\frac{1}{2}+1\right)\right] + \left[\frac{16}{3} - \frac{2}{3}\right] + 2\left[\frac{2}{3} - 0\right]$$

$$= \left[0 - \frac{3}{2}\right] + \frac{14}{3} + \frac{4}{3}$$

$$= -\frac{3}{2} + \frac{18}{3} = -\frac{3}{2} + 6 = \frac{9}{2}$$