Section 8.3

Approximate Coloring Algorithms

CHROMATIC NUMBER PROBLEM

The question of determine the chromatic number of a graph is an NP-complete problem.

CONSIDERATIONS FOR HEURISTICS

1. A vertex of high degree is harder to color than a vertex of low degree.
2. Vertices with the same neighborhood should be colored alike.
3. Coloring many vertices with the same color is a good idea.

GENERIC SEQUENTIAL COLORING ALGORITHM

Algorithm 8.2.1 Generic Sequential Coloring Algorithm.

Input: Any ordering of the vertices of a graph G.

Output: A coloring of the vertices.

Method: Use the minimum available color

1. Assign color 1 to vertex v_1.
2. If $H_{i-1} = \langle v_1, v_2, \ldots, v_{i-1} \rangle$ has been colored with j colors, then assign v_i with color k, where $k \leq j + 1$ is the minimum available color (according to some numerical ordering of the colors, say $1, 2, \ldots, n$).

LARGEST FIRST HEURISTIC

- The **largest first heuristic** orders the vertices in descending order based on their degrees.
- The vertex of highest degree is colored first, the next highest second, and so forth in a greedy manner.
- In each case, the color selected is the smallest possible legal color.
- This heuristic provides a good bound on the chromatic number of small-order graphs

A THEOREM OF WELSH AND POWELL

Theorem 8.3.1 (Welsh and Powell): Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and where $deg\ v_i \geq deg\ v_{i+1}$ for $i = 1, \ldots, n-1$. Then

$$\chi(G) \leq \max_i \min\{i, deg\ v_i + 1\}.$$

SMALLEST LAST ALGORITHM

In the **smallest last algorithm**, the vertex ordering is as follows. Let G be a graph.

- Remove the vertex of lowest degree from G and place it last on the list of vertices.
- In the subgraph that remains, select the vertex of lowest degree and place it on the list.
- Repeat until there are no more vertices.
- Color the vertices sequentially (first in-last out) in a greedy manner

COLOR DEGREE

The **color degree** of a vertex v is the number of colors used to color the vertices adjacent to v.

BRELAZ COLOR-DEGREE ALGORITHM

Algorithm 8.2.2 Brelaz Color-Degree Algorithm.

Input: A graph G.

Output: An approximate coloring of the vertices of G.

Method: Break ties based on the smallest color degree.

BRELAZ COLOR-DEGREE ALGORITHM (CONCLUDED)

1. Order the vertices in decreasing order of degrees.
2. Color the vertex of largest degree with color 1.
3. Select a vertex with maximum color-degree. If there is a tie, chose any of these vertices of largest degree in the uncolored graph.
4. Color the vertex selected in step 3 with the least possible color.
5. If all vertices are colored,
   then stop;
   else go to step 3.

A THEOREM ON ALGORITHM 8.3.2

<u>Theorem 8.3.2</u>: If G is a 2-connected bipartite graph of order at least 3, then the coloring obtained from Algorithm 8.3.2 determines the chromatic number for G.